On flatness of discrete-time systems

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Outline

Definition and problem statement

$$\begin{cases} x_{k+1} = f(x_k, u_k) \\ y_k = h(x_k, u_k) \end{cases} \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}$ is the control input, $y_k \in \mathbb{R}$ is the output, f is the state transition function, h is the output function

Definition

The system (1) *is said to be difference flat if there exists a variable y_k referred to as flat output, such that all system variables can be expressed as a function of the flat outputs and a finite number of its backward and/or forward shifts. In particular, there exist two functions $\mathcal{F}$ and $\mathcal{G}$ such that*

$$\begin{cases} x_k = \mathcal{F}(y_{k+k_{\mathcal{F}}}, \dots, y_{k+k'_{\mathcal{F}}}) \\ u_k = \mathcal{G}(y_{k+k_{\mathcal{G}}}, \dots, y_{k+k'_{\mathcal{G}}}) \end{cases} \quad (2)$$

where $k_{\mathcal{F}}$, $k'_{\mathcal{F}}$, $k_{\mathcal{G}}$ and $k'_{\mathcal{G}}$ are $\mathbb{Z}$-valued integers.

$\Rightarrow$ Characterize and/or find the flat outputs

Literature

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The LTI case [Sira-Ramirez & *al.*, 2004]

$$\left\{\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}\right. \tag{3}$$

Let the controllability matrix be defined as

$$Q_c = \left[B\ AB \cdots A^{n-1}B\right]$$

Assume that $rank(Q_c) = n$ and consider the coordinate transformation $z_k = (Q_c)^{-1}x_k$ then

$$z_{k+1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} z_k + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u_k \tag{4}$$

Then, all the states z_k^i can be expressed in terms of z_k^n (and advances)

$$\begin{aligned} z_k^{n-1} &= z_{k+1}^n + a_{n-1}z_k^n \\ z_k^{n-2} &= z_{k+1}^{n-1} + a_{n-2}z_k^n = z_{k+2}^n + a_{n-1}z_{k+1}^n + a_{n-2}z_k^n \\ &\vdots \\ z_k^1 &= z_k^2 + a_1 z_k^n = z_{k+n-1}^n + a_{n-1}z_{k+n-2}^n + \ldots + a_1 z_k^n \\ u_k &= z_{k+n}^n + a_{n-1}z_{k+n-1}^n + \ldots + a_1 z_{k+1}^n + a_0 z_k^n \end{aligned} \tag{5}$$

The LTI case (cont'd)

Proposition (Sira-Ramirez & *al.*, 2004)

A flat output of the LTI controllable system in state space form

$$x_{k+1} = Ax_k + Bu_k$$

is given, modulo a constant factor, by the linear combination of the states obtained from the last row of the inverse of the controllability matrix defined as

$$Q_c = \left[B\ AB \cdots A^{n-1}B\right]$$

Example (Euler discretization with sampling period T of a second order integrator)

$$\left\{ \begin{array}{lcl} x^1_{k+1} & = & x^1_k + Tx^2_k \\ x^2_{k+1} & = & x^2_k + Tu_k \end{array} \right. \quad (6)$$

$A = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ T \end{pmatrix}$, $Q_c = \begin{pmatrix} 0 & T^2 \\ T & T \end{pmatrix}$, $(Q_c)^{-1} = \frac{1}{T^3}\begin{pmatrix} T & -T^2 \\ -T & 0 \end{pmatrix}$

Thus, the flat output can be $y_k = -\frac{1}{T^2}x^1_k$ or $y_k = x^1_k$.

$$\begin{array}{lcl} x^1_k & = & y_k \\ x^2_k & = & \frac{1}{T}(y_{k+1} - y_k) \\ u_k & = & \frac{1}{T^2}(y_{k+2} - 2y_{k+1} + y_k) \end{array} \quad (7)$$

The LTI case: comments

- can be extended to MIMO systems
- the flat output y_k has relative degree n
- no systematic methodology to check whether a given output is flat or not (except direct approach)
- do not hold for non reversible systems ($det\ A = 0$). However,

$$\begin{cases} x^1_{k+1} &= u_k \\ x^2_{k+1} &= u_k \end{cases} \tag{8}$$

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Q_c = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ but $y_k = x^1_k$ is a flat output. Indeed

$$\begin{aligned} x^1_k &= y_k \\ x^2_k &= y_k \\ u_k &= y_{k+1} \end{aligned} \tag{9}$$

- do not hold for time-varying systems. The system

$$\begin{cases} x^1_{k+1} &= x^1_k + T_k x^2_k \\ x^2_{k+1} &= x^2_k + T_k x^2_k \end{cases} \tag{10}$$

is not flat.

Problem formulation for LPV and q-LPV systems

$$\begin{cases} x_{k+1} = A(\rho_k)x_k + B(\rho_k)u_k \\ y_k = C(\rho_k)x_k + D(\rho_k)u_k \end{cases} \quad (11)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}$ is the control input, $y_k \in \mathbb{R}$ is the output, $\rho_k = \left[\rho_k^1, \rho_k^2, \ldots, \rho_k^{L_\rho} \right] \in \Theta \subseteq \mathbb{R}^{L_\rho}$ the time varying parameter vector.

Remark: when $\rho_k = l(y_k)$, (11) is a q-LPV system

Definition

The system (11) *is said to be difference flat if, for every realization* ρ*, there exists a variable* y_k *referred to as flat output, such that all system variables can be expressed as a function of the flat outputs and a finite number of its backward and/or forward shifts. In particular, there exist two functions* $\mathcal{F}_\rho$ *and* $\mathcal{G}_\rho$ *parametrized by* ρ *such that*

$$\begin{cases} x_k = \mathcal{F}_\rho(y_{k+k_\mathcal{F}}, \ldots, y_{k+k'_\mathcal{F}}) \\ u_k = \mathcal{G}_\rho(y_{k+k_\mathcal{G}}, \ldots, y_{k+k'_\mathcal{G}}) \end{cases} \quad (12)$$

where $k_\mathcal{F}$, $k'_\mathcal{F}$, $k_\mathcal{G}$ *and* $k'_\mathcal{G}$ *are* $\mathbb{Z}$*-valued integers.*

$\Rightarrow$ Characterize and/or find the flat outputs

Example

$$\begin{cases} x_{k+1}^1 & = & x_k^1 + T_k x_k^2 \\ x_{k+1}^2 & = & \left(1 - T_k \frac{1}{2} \frac{\phi C_x S}{m} x_k^2\right) x_k^2 + T_k \frac{C_{mot}}{mR} u_k \end{cases} \tag{13}$$

When $\rho_k^1 = T_k$, $\rho_k^2 = 1 - T_k \frac{1}{2} \frac{\phi C_x S}{m} x_k^2$ and $\rho_k^3 = T_k \frac{C_{mot}}{mR}$ then (13) is an LPV system with

$$A(\rho_k) = \begin{pmatrix} 1 & \rho_k^1 \\ 0 & \rho_k^2 \end{pmatrix}, \; B(\rho_k) = \begin{pmatrix} 0 \\ \rho_k^3 \end{pmatrix}.$$

Direct approach

Consider the setting

$$
\begin{aligned}
A(\rho_k) &= \begin{pmatrix} 1 & \rho_k^1 \\ 0 & \rho_k^2 \end{pmatrix}, \\
B(\rho_k) &= \begin{pmatrix} 0 \\ \rho_k^3 \end{pmatrix}, \\
C(\rho_k) &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \\
D(\rho_k) &= 0.
\end{aligned}
\tag{14}
$$

$y_k = x_k^1$ is a flat output since the definition is satisfied.

$$
\left\{
\begin{array}{lcl}
x_k^1 & = & y_k \\
x_k^2 & = & (\rho_k^1)^{-1}(y_{k+1} - y_k) \\
u_k & = & (\rho_k^3)^{-1}\left((\rho_{k+1}^1)^{-1}(y_{k+2} - y_{k+1}) - \rho_k^2(\rho_k^1)^{-1}(y_{k+1} - y_k)\right)
\end{array}
\right.
$$

$\Rightarrow$ Alternatives ?

Relative degree

Definition

The relative degree of a dynamical system is the minimal number $r \in \mathbb{N}$ of iterations such that its output y_{k+r} at time $k+r$ is sensitive to its input u_k.

Define $\mathcal{T}^{i,j}_{\rho(k)}$ as

$$\mathcal{T}^{i,j}_{\rho(k)} = C_{\rho(k+i)} A^{\rho(k+i-1)}_{\rho(k+j+1)} B_{\rho(k+j)} \text{ if } j \leq i-1, \quad \mathcal{T}^{i,i}_{\rho(k)} = D_{\rho(k+i)} \tag{15}$$

with the transition matrix of the time-varying matrix $A(\rho_k)$

$$\begin{aligned} A^{\rho(k_1)}_{\rho(k_0)} &= A_{\rho(k_1)} A_{\rho(k_1-1)} \cdots A_{\rho(k_0)} \text{ if } k_1 \geq k_0 \\ &= \mathbf{1}_n \text{ if } k_1 < k_0 \end{aligned}$$

The relative degree r of (11) is the integer r such that

- $r = 0$ if $\mathcal{T}^{0,0}_{\rho(k)} \neq 0$ for all k
- the least integer $s < \infty$ such that for all k

$$\begin{aligned} &\mathcal{T}^{i,j}_{\rho(k)} = 0 \text{ for } i = 0, \ldots, s-1 \text{ and } j = 0, \ldots, i \\ &\mathcal{T}^{s,0}_{\rho(k)} \neq 0 \end{aligned} \tag{16}$$

$$\Rightarrow y_{k+r} = C_{\rho(k+r)} A^{\rho(k+r-1)}_{\rho(k)} x_k + \mathcal{T}^{r,0}_{\rho(k)} u_k \tag{17}$$

Inverse system approach

Theorem (Parriaux and Millérioux, SCL, 2013)

An output y_k of the system (11), with relative degree r, is a flat output if ρ_k is accessible from the output y_k and there exists a positive integer $K < \infty$ such that, the following product applies for all sequences $\{\rho_k, \ldots, \rho_{k+r+K}\} \in \Theta^{r+K+1}$ and for all $k \geq 0$:

$$P_{\rho(k+K-1:k+K-1+r)} P_{\rho(k+K-2:k+K-2+r)} \cdots P_{\rho(k:k+r)} = 0 \tag{18}$$

with

$$\begin{aligned}
P_{\rho(k:k+r)} &= A_{\rho(k)} - B_{\rho(k)} (\mathcal{T}^{r,0}_{\rho(k)})^{-1} C_{\rho(k+r)} A^{\rho(k+r-1)}_{\rho(k)} \\
\mathcal{T}^{r,0}_{\rho(k)} &= C_{\rho(k+r)} A^{\rho(k+r-1)}_{\rho(k+1)} B_{\rho(k)} \text{ if } r > 0 \\
\mathcal{T}^{0,0}_{\rho(k)} &= D_{\rho(k)} \text{ if } r = 0
\end{aligned}$$

where, for two integers k_1 and k_2, the notation $\rho(k_1 : k_2)$ explicits the dependence of the matrix P on the sequence $\{\rho_{k_1}, \rho_{k_1+1}, \ldots, \rho_{k_2}\}$.

Proof

Assume that (11) has relative degree r. Let us consider the following dynamical system (left inverse system)

$$\left\{ \begin{array}{lcl} \hat{x}_{k+r+1} & = & P_{\rho(k:k+r)}\hat{x}_{k+r} + B_{\rho(k)}(\mathcal{T}^{r,0}_{\rho(k)})^{-1} y_{k+r} \\ \hat{u}_{k+r} & = & -(\mathcal{T}^{r,0}_{\rho(k)})^{-1} C_{\rho(k+r)} A^{\rho(k+r-1)}_{\rho(k)} \hat{x}_{k+r} + (\mathcal{T}^{r,0}_{\rho(k)})^{-1} y_{k+r} \end{array} \right. \tag{19}$$

On one hand, substituting (17) ($y_{k+r} = C_{\rho(k+r)} A^{\rho(k+r-1)}_{\rho(k)} x_k + \mathcal{T}^{r,0}_{\rho(k)} u_k$) into (19) yields:

$$\hat{x}_{k+r+1} \quad = \quad P_{\rho(k:k+r)}\hat{x}_{k+r} + B_{\rho(k)}(\mathcal{T}^{r,0}_{\rho(k)})^{-1} C_{\rho(k+r)} A^{\rho(k+r-1)}_{\rho(k)} x_k + B_{\rho(k)}(\mathcal{T}^{r,0}_{\rho(k)})^{-1} \mathcal{T}^{r,0}_{\rho(k)} u_k \tag{20}$$

Noticing that $(\mathcal{T}^{r,0}_{\rho(k)})^{-1} \mathcal{T}^{r,0}_{\rho(k)} = 1$, $\epsilon_k = x_k - \hat{x}_{k+r}$ fulfills the recursion:

$$\begin{array}{lcl} \epsilon_{k+1} & = & (A_{\rho(k)} - B_{\rho(k)}(\mathcal{T}^{r,0}_{\rho(k)})^{-1} C_{\rho(k+r)} A^{\rho(k+r-1)}_{\rho(k)})\epsilon_k \\ & = & P_{\rho(k:k+r)}\epsilon_k \end{array} \tag{21}$$

Iterating (19) $K-1$ times and performing the change of variable $k+K \rightarrow k$ yields

$$x_k = P_{\rho(k-1:k-1+r)} \cdots P_{\rho(k-K:k-K+r)} x_{k-K} + \sum_{i=0}^{K-1} P^{\rho(k-1:k-1+r)}_{\rho(k-i:k-i+r)} (\mathcal{T}^{r,0}_{\rho(k-1-i)})^{-1} B_{\rho(k-i-1)} y_{k-i-1+r} \tag{22}$$

Substituting (22) into (17) gives u_k

(18) $\Leftrightarrow$ flatness

Back on the LTI case

Inverse system

$$\left\{\begin{array}{lcl} \hat{x}_{k+r+1} & = & P\hat{x}_{k+r} + B(\mathcal{T}^{r,0})^{-1}y_{k+r} \\ \hat{u}_{k+r} & = & -(\mathcal{T}^{r,0})^{-1}CA^r\hat{x}_{k+r} + (\mathcal{T}^{r,0})^{-1}y_{k+r} \end{array}\right. \tag{23}$$

Flatness condition

$$P \text{ is nilpotent with } P = A - B(\mathcal{T}^{r,0})^{-1}CA^r$$

System under consideration

$$\left\{\begin{array}{lcl} x^1_{k+1} & = & u_k \\ x^2_{k+1} & = & u_k \end{array}\right. \tag{24}$$

State space matrices

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ C = (1\ 0),\ \mathcal{T}^{1,0} = CB = 1,\ P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Flat equations

$$\begin{array}{lclcl} \hat{x}^1_{k+1} & = & x^1_k & = & y_k \\ \hat{x}^2_{k+1} & = & x^2_k & = & y_k \\ \hat{u}_{k+1} & = & u_k & = & y_{k+1} \end{array} \tag{25}$$

Nilpotent semigroup

Definition (Nilpotent semigroup)

A semigroup $\mathcal{S}$ is a set together with an associative internal law. A semigroup $\mathcal{S}$ with an absorbing element 0 is said to be nilpotent if there exists an integer $t \in \mathbb{N}^*$ such that the internal law applied to any t elements of $\mathcal{S}$ is always equal to 0. The smallest integer t is called the class of nilpotency of $\mathcal{S}$.

If $\mathcal{S}$ is a set of matrices

- the associative internal law is the matrix multiplication
- the absorbing element is the null matrix

Theorem (Parriaux and Millérioux, SCL, 2013)

If the matrix $P_{\rho(k:k+r)}$ can be triangularized independently of ρ_k, then y_k is a flat output for (11).

Proof based on Levitsky's Theorem (nilpotent semigroup $\leftrightarrow$ simultaneous triangularization)

$\Rightarrow$ allows to verify (18) without performing explicitly the product (polynomial complexity)

Example

Consider the setting

$$A(\rho_k) = \begin{pmatrix} 1 & \rho_k^1 \\ 0 & \rho_k^2 \end{pmatrix}, \quad B(\rho_k) = \begin{pmatrix} 0 \\ \rho_k^3 \end{pmatrix}, \quad C(\rho_k) = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D(\rho_k) = 0.$$

One has $D(\rho_k) = 0,\ C(\rho_{k+1})B(\rho_k) = 0$ and $C_{\rho(k+2)}A(\rho_{k+1})B(\rho_k) = \rho_{k+1}^1 \cdot \rho_k^3$. If ρ_k^1 and ρ_k^3 never vanish, the relative degree is $r = 2$.

$P_{\rho(k:k+1)} = A(\rho_k) - B(\rho_k)C(\rho_{k+1})A(\rho_{k+1})A(\rho_k)$ reads

$$P_{\rho(k:k+1)} = \begin{pmatrix} 1 & \rho_k^1 \\ -(\rho_{k+1}^1)^{-1} & -(\rho_{k+1}^1)^{-1}\rho_k^1 \end{pmatrix}$$

$\Rightarrow$ (18) is fulfilled since $P_{\rho(k:k+2)}P_{\rho(k:k+1)} = 0$ for any realization of ρ_k but does not generate a nilpotent semigroup (sufficient condition)

Structured linear systems

$$\Sigma_\rho : \; x_{k+1} = Ax_k + Bu_k \tag{26}$$

where

- $x_k \in \mathbb{R}^n$ is the state vector
- $u_k \in \mathbb{R}$ is the input
- Only the sparsity pattern of the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$ is known ('0' or '1' entries)

$\Rightarrow$ To the '1' entries are assigned the time-varying parameter ρ_k^i of the LPV system (11)

$\Rightarrow$ generic flatness for the LPV system $\leftrightarrow$ structural flatness of the system Σ_ρ

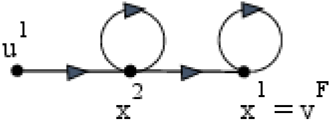
Digraph

A digraph associated to Σ_ρ is a couple $(\mathcal{V}, \mathcal{E})$ where

- $\mathcal{V}$ is the vertex set associated to Σ_ρ
- $\mathcal{E}$ is the edge set associated to Σ_ρ

Example:

$$A(\rho_k) = \begin{pmatrix} 1 & \rho_k^1 \\ 0 & \rho_k^2 \end{pmatrix}, \; B(\rho_k) = \begin{pmatrix} 0 \\ \rho_k^3 \end{pmatrix},$$
$$C(\rho_k) = \begin{pmatrix} 1 & 0 \end{pmatrix}, D(\rho_k) = 0.$$



Flat output characterization

$$\Sigma_\rho : \; x_{k+1} = Ax_k + Bu_k$$

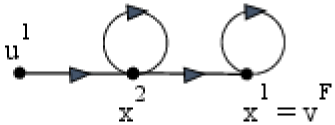
Theorem (Millérioux and Boukhobza, CDC, 2015)

The output $y_k = x_k^i$ $(i = 1, \ldots, n)$ or $y_k = u_k$ of the LPV system (11) is generically a flat output iff, in the associated digraph $\mathcal{G}(\Sigma_\rho)$, all the following conditions hold:
C0. $\mathbf{v^F}$ *is a successor of* $\mathbf{u}$*;*
C1. *The lengths of all the* $\{\mathbf{u}\}$*-*$\{\mathbf{v^F}\}$ *simple paths are identical and equal to* $\ell(\mathbf{u}, \mathbf{v^F})$*;*
C2. *All the cycles cover at least an element of* $V_{ess}(\boldsymbol{U}, \{\mathbf{v^F}\})$*.*

Examples

Example 1:

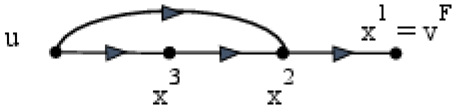
- Condition **C2** is not fulfilled with $y_k = \mathbf{x_k^2}$
- Conditions **C0**, **C1** and **C2** are satisfied with $y_k = \mathbf{x_k^1}$



Examples

Example 2:

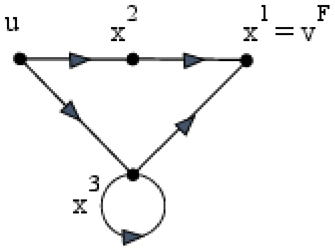
- Condition **C0** is satisfied with $y_k = \mathbf{x_k^1}$
- Condition **C1** is not satisfied with $y_k = \mathbf{x_k^1}$



Examples

Example 3:

- Condition **C0** is satisfied with $y_k = \mathbf{x_k^1}$
- Condition **C1** is satisfied with $y_k = \mathbf{x_k^1}$
- Condition **C2** is not satisfied with $y_k = \mathbf{x_k^1}$



Connection between the state-space and the graph approaches

Proposition

The relative degree r of the system (11) is equal to $\ell(\mathbf{u}, \mathbf{v}^{\mathbf{F}})$ of $\mathcal{G}(\Sigma_\rho)$.

Proposition

The integer K is equal to the maximal path length over all the simple $\{\mathbf{u}\}$-$\mathbf{X}$ paths of $\mathcal{G}(\Sigma_\rho)$.

Conclusion

Flat output characterization of switched linear, LPV and q-LPV discrete-time systems

- Alternatives to the direct approach
- Inverse system approach: hold for MIMO systems, exponential complexity
- Nilpotent semigroups approach: holds for MIMO systems, polynomial complexity but more conservative (sufficient condition)
- Graph-oriented approach: generic flatness, polynomial complexity, can be extended to MIMO systems, allows to find flat outputs
- Mixed state-space and graph-based approaches: a complete framework
- Application to the construction of ciphers in cryptography

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