Observability and observer design for linear TDS

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Outline

- Motivation
- Part I : Observability analysis
- Part II : Observer design
- Simulations
- Conclusions

Motivation

- Goal : Using available measurements to estimate the internal states ;
- Reason : Controller needs sometimes full/partial states, or sometimes we need to filter the noisy measurement
- This topic has been widely studied in the literature for different types of systems :
  - Linear system (Kalman et al. (1969); Silverman (1969); Molinari (1976); Bhattacharyya (1978); Hautus (1983); Darouach et al. (1994); Yang and Wilde (1988); Hou and Muller (1992); Kudva et al. (1980); Wang et al. (1975); Hostetter and Meditch (1973)...),
  - Nonlinear system (Hermann and Krener (1977); Hirschorn (1979); Krener (1985); Respondek (1990); Diop and Fliess (1991); Isidori (1995); Gauthier et al. (1992). ; Boutat et al. (2001) ; Barbot et al. (2005) ; Hammouri et al. (1994) ; Farza (2005)...),
  - Time-delay (Bhat and Koivo (1976); Fattouh et al. (1999a); Germani et al. (1998); Hou et al. (2002); Conte et al. (2003); Fliess and Mounier (1998); Sename (2001); Fu et al. (2004); Darouach (2006a)...)
  - ...

Observability VS Observer

Given a general nonlinear system :

- First question : It is possible to estimate full/partial state ? (Analyze Observability/Detectability)
  - depending on the known information (output, input, parameters,...)
  - depending as well on the system (with/without delay, singular,...)
- Second question : If yes, then how to estimate them ? (Design Differentiator/Observer)
  - Differentiator : Algebraic, HOSM, Homogeneity, High-gain,...
  - Observer : Luenberger, Kalman, High-gain, adaptive, sliding mode,...

They are not equivalent !

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Today's topic

For a general linear time-delay systems :

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{k_a} A_i x(t-\tau_i) + \sum_{i=0}^{k_b} B_i u(t-\tau_i)\\ y(t) &= \sum_{i=0}^{k_c} C_i x(t-\tau_i) + \sum_{i=0}^{k_d} D_i u(t-\tau_i)\end{aligned}$$

we are going to answer :

- under which condition the system is observable (backward and forward) ;
- under which condition the state can be estimate via a simple Luenberger-like observer ;

Part I : Observability analysis

Some relevant existing works

- Known input :
  - one delay + only in the state (Fattouh et al. (1999b); Darouach (2001)...)
  - multiple delays both in the state and output (Emre and Khargonekar (1982); Sename (1997); Hou and Muller (1992)...)
- Unknown input :
  - one delay + only in the state (Conte et al. (2003); Sename (2001); Darouach (2006b)...)
  - multiple delays both in the state and output (Not yet solved !)

Assumption

Considered system :

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{k_a} A_i x(t-\tau_i) + \sum_{i=0}^{k_b} B_i u(t-\tau_i) \\ y(t) &= \sum_{i=0}^{k_c} C_i x(t-\tau_i) + \sum_{i=0}^{k_d} D_i u(t-\tau_i)\end{aligned}$$

- Assumptions :
  - delay is commensurate delay, i.e. $\exists h$ s.t. $\tau_i = ih$;
  - input is unknown ;

The studied can be rewritten as :

$$\begin{cases}\dot{x}(t) = \sum_{i=0}^{k_a} A_i x(t-ih) + \sum_{i=0}^{k_b} B_i u(t-ih) \\ y(t) = \sum_{i=0}^{k_c} C_i x(t-ih) + \sum_{i=0}^{k_e} D_i u(t-ih)\end{cases} \tag{1}$$

Delay operator

Delay operator $\delta : x(t) \to x(t-h)$ with $\delta^k x(t) = x(t-kh)$, $k \in \mathbb{N}_0$;

$\mathbb{R}[\delta]$: the polynomial ring of δ over the field $\mathbb{R}$;

i.e. for any $a(\delta) = \sum_{i=0}^{d_a} a_i \delta^i \in \mathbb{R}[\delta]$ with $a_i \in \mathbb{R}$:

$$a(\delta) + b(\delta) = \sum_{i=0}^{\max\{d_a, d_b\}} (a_i + b_i)\delta^i$$
$$a(\delta)b(\delta) = \sum_{i=0}^{d_a} \sum_{j=0}^{d_b} a_i b_j \delta^{i+j}$$

it means

$$a(\delta) + b(\delta) = b(\delta) + a(\delta)$$

and

$$a(\delta)b(\delta) = b(\delta)a(\delta)$$

$\mathbb{R}[\delta]$ *is a commutative ring!*

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LTDS over $\mathbb{R}[\delta]$

Consider the system

$$\begin{cases} \dot{x}(t) & = & \sum_{i=0}^{k_a} A_i x(t-ih) + \sum_{i=0}^{k_b} B_i u(t-ih) \\ y(t) & = & \sum_{i=0}^{k_c} C_i x(t-ih) + \sum_{i=0}^{k_e} D_i u(t-ih) \end{cases}$$

then we have :

$$\begin{cases} \dot{x}(t) & = & A(\delta)x(t) + B(\delta)u(t) \\ y(t) & = & C(\delta)x(t) + D(\delta)u(t) \end{cases} \tag{2}$$

with $A(\delta) := \sum_{i=0}^{k_a} A_i \delta^i$, $B(\delta) := \sum_{i=0}^{k_b} B_i \delta^i$, $C(\delta) := \sum_{i=0}^{k_c} C_i \delta^i$, $D(\delta) := \sum_{i=0}^{k_d} D_i \delta^i$.

$A(\delta)$, $B(\delta)$, $C(\delta)$, and $D(\delta)$ are matrices over the polynomial ring $\mathbb{R}[\delta]$!

What is the interest to introduce δ ?

LTI system

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

with A, B, C and D being matrices over $\mathbb{R}$.

LTD system

$$\begin{cases} \dot{x} &= A(\delta)x + B(\delta)u \\ y &= C(\delta)x + D(\delta)u \end{cases}$$

with $A(\delta)$, $B(\delta)$, $C(\delta)$, *and* $D(\delta)$ *being matrices over* $\mathbb{R}[\delta]$.

- Interest : The same structure of the systems implies the existing results (Observability/Observer) for LTI MIGHT be applicable for LTD !
- Attention : The different definition of the matrices implies particular attentions SHOULD be paid when adapting those results !

Dependence of delay

Since TDS contains delay, the observability depends as well on delay (*backward or forward*) !

Definition

[BUIO] System (1) is **backward UIO (BUIO)** on $[t_1, t_2]$ if for each $\tau \in [t_1, t_2]$ there exist $t_1' < t_2' \le \tau$ such that, for all u and φ,

$$y(t;\varphi,u) = 0 \text{ for all } t \in [t_1', t_2'] \text{ implies } x(\tau;\varphi,u) = 0.$$

Definition

[FUIO] System (1) is **forward UIO (FUIO)** on $[t_1, t_2]$ if for each $\tau \in [t_1, t_2]$ there exist $t_2' > t_1' \ge \tau$ such that, for all u and φ,

$$y(t;\varphi,u) = 0 \text{ for all } t \in [t_1', t_2'] \text{ implies } x(\tau;\varphi,u) = 0.$$

Remarks

- These definitions are essentially formulated following the observability definitions given in Kalman et al. (1969) and the strong observability given in Hautus (1983) for LTI ;
- UIO : reconstruction using past, actual, and future values of y ;
- FUIO : reconstruction using actual and future values of y ;
- BUIO : reconstruction using past and actual values of y ;
- Either BUIO or FUIO implies UIO ;
- BUIO and FUIO do not exclude each other.

Ex :

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + \delta x_2 ; \quad y_1 = \delta x_1, \quad y_2 = x_2$$

is BUIO and FUIO.

Recall LTI

Linear TI system

$$\begin{cases} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{cases}$$

NS condition of strong observability (Hautus (1983)Trentelman et al. (2001)) :

$$\text{rank} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = n + \text{rank} \begin{bmatrix} B \\ D \end{bmatrix} \text{ for all } s \in \mathbb{C} \tag{3}$$

Easily to check, but sometimes we need more (ex : the relation) !

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Recall LTI

LTI system

$$\begin{cases} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{cases}$$

Iterative algorithm (Molinari (1976); Silverman (1969)) :

$$\Delta_0 \triangleq 0, \quad G_0 \triangleq C, \quad F_0(\delta) \triangleq D, \; N_0 = 0$$
$$\begin{bmatrix} F_{k+1} & G_{k+1} \\ 0 & \Delta_{k+1} \end{bmatrix} \triangleq T_k \begin{bmatrix} \Delta_k B & \Delta_k A \\ F_k & G_k \end{bmatrix}$$
$$N_{k+1} \triangleq \begin{bmatrix} N_k \\ \Delta_{k+1} \end{bmatrix}$$

then system is SO iff $\exists k \in \mathbb{N}_0$, s.t. $\text{rank} N_{k+1} = \text{rank} N_k = n$.

Difficulties

Difficulties from LTI to LTDS (Inverse problem) :

- over $\mathbb{R}$, rank$N_k = n$ implies the existence of the left inverse of N_k ;
- over $\mathbb{R}[\delta]$, it is not trivial ;

ex :

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \text{VS} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & \delta \end{bmatrix} \text{VS} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & \delta \end{bmatrix}$$

Definitions

Definition

A given polynomial matrix $A(\delta) \in \mathbb{R}^{n\times q}[\delta]$ is said to be left (or right) unimodular (or invertible) over $\mathbb{R}[\delta]$ if there exists $A_L^{-1}(\delta) \in \mathbb{R}^{q\times n}[\delta]$ with $n \geq q$ (or $A_R^{-1}(\delta) \in \mathbb{R}^{q\times n}[\delta]$ with $n \leq q$), such that $A_L^{-1}(\delta)A(\delta) = \mathrm{I}_q$ (or $A(\delta)A_R^{-1}(\delta) = \mathrm{I}_n$). A square matrix $A(\delta) \in \mathbb{R}^{n\times n}[\delta]$ is said to be unimodular (or invertible) over $\mathbb{R}[\delta]$ if $A_L^{-1}(\delta) = A_R^{-1}(\delta)$.

Properties : Hermite form

For $P(\delta)$ be a matrix of $q \times s$ dimension with rank equal to r (clearly $r \leq \min\{q, s\}$), there exists an invertible matrix $T(\delta)$ over $\mathbb{R}[\delta]$ such that $P(\delta)$ is put into (column) Hermite form :

$$T(\delta)P(\delta) = \left[\begin{array}{c} P_1(\delta) \\ 0 \end{array} \right]$$

where $P_1(\delta)$ is of $r \times s$ dimension, and rank$P_1 = r$.

Properties : Smith form

For $P(\delta)$ be a matrix of $q \times s$ dimension with rank equal to r , there exist two invertible matrices $U(\delta)$ and $V(\delta)$ over $\mathbb{R}[\delta]$ such that $P(\delta)$ is reduced to its Smith form :

$$U(\delta)P(\delta)V(\delta) = \begin{bmatrix} \mathrm{diag}(\psi_1(\delta)\cdots\psi_r(\delta)) & 0 \\ 0 & 0 \end{bmatrix}$$

where$\{\psi_i(\delta)\}$'s are called invariant factors, noted as :

$$\mathrm{Inv}_S[P(\delta)] = \{\psi_i(\delta)\}_{1\leq i\leq r}$$

Conditions on invertibility

Lemma

A polynomial matrix $D(\delta) \in \mathbb{R}^{p\times m}[\delta]$ *is left (or right) unimodular (or invertible) over* $\mathbb{R}[\delta]$ *if and only if the following conditions are satisfied :*

1. $\mathrm{rank}_{\mathbb{R}[\delta]} D(\delta) = m \leq p$ *(or* $\mathrm{rank}_{\mathbb{R}[\delta]} D(\delta) = p \leq m$ *);*
2. $\mathrm{Inv}_S\,[D(\delta)] \subset \mathbb{R}$.

It is said to be unimodular (or invertible) over $\mathbb{R}[\delta]$ *if and only if the following conditions are satisfied :*

1. $\mathrm{rank}_{\mathbb{R}[\delta]} D(\delta) = p = m$ *;*
2. $\mathrm{Inv}_S\,[D(\delta)] \subset \mathbb{R}$.

Following the ideas of Silverman (1969) and Molinari (1976), define $\{\Delta_k(\delta)\}$ matrices generated by the following algorithm :

$$\Delta_0 \triangleq 0, \quad G_0(\delta) \triangleq C(\delta), \quad F_0(\delta) \triangleq D(\delta)$$
$$\begin{bmatrix} F_{k+1}(\delta) & G_{k+1}(\delta) \\ 0 & \Delta_{k+1}(\delta) \end{bmatrix} \triangleq T_k(\delta) \begin{bmatrix} \Delta_k(\delta) B(\delta) & \Delta_k(\delta) A(\delta) \\ F_k(\delta) & G_k(\delta) \end{bmatrix} \tag{4}$$

and

$$M_0(\delta) \triangleq N_0(\delta) \triangleq \Delta_0, \; N_{k+1}(\delta) \triangleq \begin{bmatrix} N_k(\delta) \\ \Delta_{k+1}(\delta) \end{bmatrix}, \quad \text{for } k \geq 0$$
$$\begin{bmatrix} M_{k+1}(\delta) \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} \operatorname{diag}\left(\psi_1^{k+1}(\delta), \ldots, \psi_{r_{k+1}}^{k+1}(\delta)\right) & 0 \\ 0 & 0 \end{bmatrix} = U_{k+1}(\delta) N_{k+1}(\delta) V_{k+1}(\delta) \tag{5}$$

where $M_{k+1}(\delta)$ is the Smith form of $N_{k+1}(\delta)$.

Theorem

Lemma

$\{M_k\}$ matrices given by (4)-(5) are independent of the choice of $\{T_k\}$, $\{U_k\}$, and $\{V_k\}$.

Lemma

$M_{k+1}(\delta) = M_k(\delta)$ if, and only if, $\Delta_{k+1}(\delta) = P(\delta) N_k(\delta)$ for some matrix $P(\delta)$.

Theorem

If $M_{k+1}(\delta) = M_k(\delta)$, then $M_{k+i}(\delta) = M_k(\delta)$ for all $i \geq 0$.

Theorem

Theorem

After a finite number of steps, let's say k^*, the algorithm (4)-(5) converges, i.e., there exists a least integer k^* such that $M_{k^*+1}(\delta) = M_{k^*}(\delta)$. Furthermore, k^* is independent of the choice of $\{T_k\}$, $\{U_k\}$, and $\{V_k\}$ matrices used in (4)-(5).

Remark

Theorem 8 gives a way to have an upper estimation of how many steps will pass before the rank of M_k does change, if it does.

Corollary

Let k^* be the least integer k such that $M_{k+1} = M_k$, then for all $i \geq 0$, $M_{k^*+i} = M_{k^*}$.

What is the role of M_{k^*} ?

LTD system

$$\begin{cases} \dot{x} & = & A(\delta)x + B(\delta)u \\ y & = & C(\delta)x + D(\delta)u \end{cases}$$

Following the algorithm :

$$\begin{bmatrix} F_{k+1}(\delta) & G_{k+1}(\delta) \\ 0 & \Delta_{k+1}(\delta) \end{bmatrix} \triangleq T_k(\delta) \begin{bmatrix} \Delta_k(\delta)B(\delta) & \Delta_k(\delta)A(\delta) \\ F_k(\delta) & G_k(\delta) \end{bmatrix}$$

finally $\exists t^*$ such that

$$N_{k^*}(\delta)x(t) = Y(t) \text{ for all } t \geq t^* \tag{6}$$

where $Y(t)$ is the vector containing only the derivatives of the y and its delay, and

$$M_{k^*}(\delta) = U_{k^*}(\delta) N_{k^*}(\delta) V_{k^*}(\delta)$$

where $M_{k^*}(\delta)$ is the Smith form of $N_{k^*}(\delta)$.

UI Observability

Theorem

System is UIO, if $M_{k^*}(\delta)$ has n invariant factors of the form $a\delta^j$, where $a \in \mathbb{R}_{\neq 0}$, $j \in \mathbb{N}_0$. The formula to reconstruct $x(t)$ is

$$x(t) = \begin{bmatrix} V_{k^*} M_{k^*}^{-1} & 0 \end{bmatrix} U_{k^*} Y(t), \; t > t_1^* \tag{7}$$

Furthermore, the i-th entry of $x(t)$ is given by an expression of the form :

$$x_i(t) = \sum_{k,j} q_{k,j} y_k^{(j)}(t) \tag{8}$$

where $y_k^{(i)}(t)$ is the i-th derivative of the k-th entry of $y(t)$ and $0 \neq q_{j,k}^i \in \mathbb{R}\left[\delta, \delta^{-1}\right]$.

It is obvious since $Y(t) = N_{k^*}(\delta)x(t)$ (for all $t \geq t^*$), thus $\exists t_1^*$ such that $x(t) = \begin{bmatrix} V_{k^*} M_{k^*}^{-1} & 0 \end{bmatrix} U_{k^*} Y(t), \; t > t_1^* \ldots$

BUI & FUI Observability

Theorem

System is BUIO, if system is UIO, and one of the following equivalent conditions are satisfied :

1. $q_{j,i}$ *of (8) belongs to* $\mathbb{R}[\delta]$ *;*
2. $M_{k^*}(\delta)$ *has n invariant factors belonging to* $\mathbb{R}$ *;*

The first condition is evident since $x_i(t) = \sum_{k,j} q_{k,j} y_k^{(j)}(t)$, and the second is due to the fact that $x(t) = \begin{bmatrix} V_{k^*} M_{k^*}^{-1} & 0 \end{bmatrix} U_{k^*} Y(t)$.

Remark

For the case $h = 0$, *the condition that* M_{k^*} *is invertible over* $\mathbb{R}$ *is also NSC for the system to be strong observability (Molinari (1976))*

Theorem

System is BUIO, if system is UIO, and every polynomial $q_{j,i}$ *of (8) belongs to* $\mathbb{R}\left[\delta^{-1}\right]$.

Example 1

Let us consider the following example :

$$A(\delta) = \begin{pmatrix} 1 & \delta & \delta & 0 & 0 \\ -\delta^2 & 0 & \delta & 0 & -\delta \\ \delta & 1 & -\delta^2 & -1+\delta & 1-\delta+\delta^3 \\ 0 & 0 & -1 & 0 & 0 \\ \delta-\delta^2 & 0 & -1+\delta & 2 & 1-\delta \end{pmatrix}$$

$$B(\delta) = \begin{pmatrix} -\delta & 0 \\ 0 & -\delta \\ 1+\delta & 1 \\ 1 & 0 \\ 0 & 1-\delta \end{pmatrix}$$

$$C(\delta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta \end{pmatrix}, \quad D(\delta) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1+\delta & 0 \end{pmatrix}$$

Example 1

After applying the algorithm, $M_3 = M_4 = \mathrm{diag}\left(1, 1, \delta, \delta, \delta^2\right)$. Thus, in this case $k^* = 3$, and the system is UIO. Explicitly, we have that the state vector can be expressed as

$$
\begin{aligned}
x_1 &= y_1 \\
x_2 &= -\delta^{-1} y_1 + \delta^{-1}\dot{y}_1 + \frac{1}{2}\left(1-\delta^{-1}\right)\dot{y}_2 - \frac{1}{2}\delta^{-1}\dot{y}_3 \\
x_3 &= \delta^{-1} y_1 + y_2 - \delta^{-1}\dot{y}_1 - \frac{1}{2}\left(1-\delta^{-1}\right)\dot{y}_2 + \frac{1}{2}\delta^{-1}\dot{y}_3 \\
&\quad -\frac{1}{2}\left(1-\delta^{-1}\right)\ddot{y}_2 + \frac{1}{2}\delta^{-1}\ddot{y}_3 \\
x_4 &= \frac{1}{2}\left(1-\delta^{-1}\right)\dot{y}_2 - \frac{1}{2}\delta^{-1}\dot{y}_3 \\
x_5 &= \left(\delta^{-2}-\delta^{-1}\right) y_1 + \left(\delta^{-1}-1\right) y_2 + \delta^{-1} y_3 \\
&\quad + \left(\delta^{-1}-\delta^{-2}\right)\dot{y}_1 + \frac{1}{2}\left(1-\delta^{-1}\right)^2\dot{y}_2 + \frac{1}{2}\left(\delta^{-2}-\delta^{-1}\right)\dot{y}_3
\end{aligned}
$$

Therefore, the system **is FUIO**.

Example 2

let us consider

$$\begin{aligned} A &= \begin{pmatrix} 0 & -1 & 1 \\ -1 & \delta & 0 \\ 1 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \delta \end{pmatrix} \\ C &= \begin{pmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; D = \begin{pmatrix} -2\delta & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

In this example, we have $M_2 = M_3 = \operatorname{diag}(1,1,1)$. It is UIO, and **BUIO**. Indeed, it is easy to verify that the state variables can be expressed as

$$\begin{aligned} x_1(t) &= -\dot{y}_2(t) + \delta y_2(t), \; x_2(t) = y_2(t) \\ x_3(t) &= -\ddot{y}_2(t) + \delta \dot{y}_2(t) + y_2(t) \end{aligned}$$

Part II : Observer design

Problem statement

Consider LTDS :

$$\begin{cases} \dot{x} &= A(\delta)x + B(\delta)u \\ y &= C(\delta)x + D(\delta)u \end{cases}$$

where $x(t) \in \mathbb{R}^{n_x}$, $y(t) \in \mathbb{R}^p$, and $u(t) \in \mathbb{R}^m$.

Assumption

For the studied system, there exists a least integer $k^* \in \mathbb{N}_0$ such that $\mathrm{rank}_{\mathbb{R}[\delta]} M_{k^*}(\delta) = n_x$, and $M_{k^*}(\delta)$ is unimodular over $\mathbb{R}[\delta]$.

The above assumption implies that the system is BUIO. It means :

1. the iterative algorithm yields $x_i(t) = \sum\limits_{k,j} q_{k,j} y_k^{(j)}(t)$;
2. we can use the differentiator to estimate the state.

Question : it is possible to use the basic observer ?

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Recall LTI

Consider LTI system :

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

there exists a Luenberger-like observer (Trentelman et al. (2001)) :

$$\dot{\xi} = P\xi + Qy$$
$$\hat{x} = \xi + Ky$$

if and only if

$$\operatorname{rank} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = n + \operatorname{rank} \begin{bmatrix} B \\ D \end{bmatrix} \text{ for all } s \in \mathbb{C} \tag{9}$$

and

$$\operatorname{rank} \begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B \\ D \end{bmatrix} + \operatorname{rank} D \tag{10}$$

Need another condition

Remark

When treating linear systems without delay, the conditions imposed in Assumption 1 is equivalent to :

$$\operatorname{rank}\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = n + \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix} \text{ for all } s \in \mathbb{C} \tag{11}$$

We need to find another condition which should be equivalent to the following one (for the case $h=0$) :

$$\operatorname{rank}\begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix} + \operatorname{rank} D$$

Need another condition

Remark

When treating linear systems without delay, the conditions imposed in Assumption 1 is equivalent to :

$$\operatorname{rank}\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = n+\operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix} \textit{ for all } s\in\mathbb{C} \tag{11}$$

We need to find another condition which should be equivalent to the following one (for the case $h=0$) :

$$\operatorname{rank}\begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix} + \operatorname{rank}D$$

Proposed the second condition

Assumption

It is assumed that

$$\mathrm{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} = \mathrm{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} \tag{12}$$

Remark

When treating linear systems without delay, the condition imposed in Assumption 2 is equivalent to :

$$\mathrm{rank}\begin{bmatrix} CB & D \\ D & 0 \\ B & 0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} B \\ D \end{bmatrix} + \mathrm{rank} D$$

Lemma

Lemma

Suppose Assumption 2 is satisfied, then there exists a matrix $W(\delta)$ *satisfying the following conditions :*

1. $W(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix};$
2. *for any matrix* $J(\delta)$ *such that* $J(\delta)\begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} = 0$*, then* $J(\delta)W(\delta) = 0$*.*

The first condition is obvious since

$$\mathrm{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} = \mathrm{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix}.$$

Key point

Transform into Hermite form :

$$U(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} V(\delta) \\ 0 \end{bmatrix}$$

Since $\exists W(\delta)$ s.t. $W(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$, then $\exists \bar{V}(\delta)$ s.t. $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$. Finally select $W(\delta) = [\bar{V}(\delta), 0]U(\delta)$.

As$V(\delta)$ is full row rank over $\mathbb{R}[\delta]$, then for any matrix $J(\delta)$ such that $J(\delta)\begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} = 0$ we always have $J(\delta)\bar{V}(\delta) = 0$, which implies $J(\delta)W(\delta) = 0$.

System decomposition

For LTDS

$$\begin{cases} \dot{x} & = & A(\delta)x + B(\delta)u \\ y & = & C(\delta)x + D(\delta)u \end{cases}$$

decompose $W(\delta) = [W_1(\delta), W_2(\delta)]$, then we have $W_1(\delta)D(\delta) = 0$. Thus, we obtain

$$W_1(\delta)y = W_1(\delta)C(\delta)x$$

which yields

$$W_1(\delta)\dot{y} = W_1(\delta)C(\delta)A(\delta)x + W_1(\delta)C(\delta)B(\delta)u$$

System decomposition

Since $W(\delta)\begin{bmatrix} C(\delta)B(\delta) \\ D(\delta) \end{bmatrix} = \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix}$, then we have

$$\begin{aligned} W_1(\delta)\dot{y} + W_2(\delta)y &= W(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} x + W(\delta)\begin{bmatrix} C(\delta)B(\delta) \\ D(\delta) \end{bmatrix} u \\ &= W(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} x + \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u \end{aligned}$$

Thus $\begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u = W(\delta)\begin{bmatrix} \dot{y} - C(\delta)A(\delta) \\ y - C(\delta) \end{bmatrix}$

System decomposition

Decompose again the matrix $W(\delta)$ as

$$W(\delta) = [W_1(\delta), W_2(\delta)] = \begin{pmatrix} K(\delta) \\ \Gamma(\delta) \end{pmatrix} = \begin{bmatrix} K_1(\delta) & K_2(\delta) \\ \Gamma_1(\delta) & \Gamma_2(\delta) \end{bmatrix}$$

since

$$\begin{array}{rcl} \dot{x} & = & A(\delta)x + B(\delta)u \\ y & = & C(\delta)x + D(\delta)u \end{array}, \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u = W(\delta) \begin{bmatrix} \dot{y} - C(\delta)A(\delta) \\ y - C(\delta) \end{bmatrix}$$

then system (2) becomes

$$\begin{array}{rl} \dot{x} & = \bar{A}(\delta)x + K_1(\delta)\dot{y} + K_2(\delta)y \\ y & = \tilde{C}(\delta)x + \Gamma_1(\delta)\dot{y} + \Gamma_2(\delta)y \end{array} \tag{13}$$

where $\bar{A}(\delta) = A - K \begin{bmatrix} CA \\ C \end{bmatrix}$ and $\tilde{C}(\delta) = C - \Gamma \begin{bmatrix} CA \\ C \end{bmatrix}$.

System decomposition

Suppose $\mathrm{rank}_{\mathbb{R}[\delta]}\tilde{C}(\delta) = r \leq p$, then $\exists \Lambda(\delta)$ s.t. $\Lambda(\delta)\tilde{C}(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix}$. By noting $\bar{y} = \Lambda(\delta)y$, finally system (13) can be decomposed into :

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x + K_1(\delta)\Lambda^{-1}(\delta)\dot{\bar{y}} + K_2(\delta)\Lambda^{-1}(\delta)\bar{y} \\ \bar{y} &= \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix} x + \bar{\Gamma}_1(\delta)\dot{\bar{y}} + \bar{\Gamma}_2(\delta)\bar{y} \end{aligned} \tag{14}$$

where

$$\begin{aligned} \bar{\Gamma}_1(\delta) &= \Lambda(\delta)\Gamma_1(\delta)\Lambda^{-1}(\delta) \\ \bar{\Gamma}_2(\delta) &= \Lambda(\delta)\Gamma_2(\delta)\Lambda^{-1}(\delta) \end{aligned} \tag{15}$$

Observer normal form

Define the following polynomial matrix over $\mathbb{R}[\delta]$:

$$\bar{\mathscr{O}}_l(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ \bar{C}(\delta)\bar{A}(\delta) \\ \vdots \\ \bar{C}(\delta)\bar{A}^{l-1}(\delta) \end{bmatrix} \in \mathbb{R}^{rl\times n_x}[\delta] \tag{16}$$

where $l \in \mathbb{N}_0$, and let us recall a useful result stated in Hou and Muller (1992).

Theorem of Hou and Muller (1992)

There exists a bicausal generalized change of coordinates $z = T(\delta)x$ which transforms the following system :

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x \\ \bar{y} &= \bar{C}(\delta)x \end{aligned} \tag{17}$$

with $\mathrm{rank}_{\mathbb{R}[\delta]}\bar{C}(\delta) = r$ into the following observer normal form :

$$\begin{cases} \dot{z} = A_0 z + F(\delta)\bar{y} \\ \bar{y} = C_0 z \end{cases}$$

where $F(\delta) = \left[F_1^T(\delta), \cdots, F_{l^*}^T(\delta)\right]^T$ and

$$A_0 = \begin{bmatrix} 0 & \mathrm{I}_r & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{I}_r \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{rl^* \times rl^*}, C_0 = [\mathrm{I}_r, 0, \cdots, 0] \in \mathbb{R}^{r \times rl^*} \tag{18}$$

if and only if there exists a least integer $l^* \in \mathbb{N}_0$ such that $\bar{\mathscr{O}}_{l^*}(\delta)$ defined in (16) is left unimodular over $\mathbb{R}[\delta]$.

Theorem of Hou and Muller (1992)

Moreover, the bicausal generalized change of coordinates $z = T(\delta)x$ with $T(\delta) = \mathrm{col}\{T_1(\delta), \cdots, T_{l^*}(\delta)\}$ is defined as follows :

$$\begin{cases} T_1(\delta) = \bar{C}(\delta) \\ T_{i+1}(\delta) = T_i(\delta)\bar{A}(\delta) - F_i(\delta)\bar{C}(\delta), \text{ for } 1 \leq i \leq l^* - 1 \end{cases} \tag{19}$$

with $F_i(\delta)$ being determined through the following equations :

$$[F_{l^*}(\delta), \cdots, F_1(\delta)] = \bar{C}(\delta)\bar{A}^{l^*}(\delta)\left[\bar{\mathscr{O}}_{l^*}(\delta)\right]_L^{-1} \tag{20}$$

Observer normal form

Lemma

If Assumption 1 is satisfied for the quadruple $(A(\delta), B(\delta), C(\delta), D(\delta))$ *in system (2), then there exists a least integer* $l^* \in \mathbb{N}_0$ *such that*

$$\mathscr{O}_{l^*}(\delta) = \begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \\ \vdots \\ C(\delta)A^{l^*-1}(\delta) \end{bmatrix} \in \mathbb{R}^{pl^* \times n_x}[\delta] \tag{21}$$

is left unimodular over $\mathbb{R}[\delta]$.

Observer normal form

Lemma

If Assumption 1 is satisfied for the quadruple $(A(\delta), B(\delta), C(\delta), D(\delta))$ *defined in (2), then for the deduced system (13) there exists a least integer* $l^* \in \mathbb{N}_0$ *such that*

$$\tilde{\mathscr{O}}_{l^*}(\delta) = \begin{bmatrix} \tilde{C}(\delta) \\ \tilde{C}(\delta)\bar{A}(\delta) \\ \vdots \\ \tilde{C}(\delta)\bar{A}^{l^*-1}(\delta) \end{bmatrix} \in \mathbb{R}^{pl^* \times n_x}[\delta] \qquad (22)$$

is left unimodular over $\mathbb{R}[\delta]$.

Observer normal form

Lemma

If there exists a least integer $l^* \in \mathbb{N}_0$ *such that* $\tilde{\mathscr{O}}_{l^*}(\delta)$ *defined in (22) is left unimodular over* $\mathbb{R}[\delta]$, *then* $\mathscr{O}_{l^*}(\delta)$ *defined in (16) is left unimodular over* $\mathbb{R}[\delta]$.

Theorem

If Assumption 1 and Assumption 2 are both satisfied for system (2), then for the deduced system (14) there exists a least integer $l^* \in \mathbb{N}_0$ *such that* $\mathscr{O}_{l^*}(\delta)$ *defined in (16) is left unimodular over* $\mathbb{R}[\delta]$.

Observer normal form

Corollary

If Assumption 1 and Assumption 2 are both satisfied for system (2), then there exists a bicausal generalized change of coordinates $z = T(\delta)x$ defined in (19) such that system (14) can be transformed into :

$$\left\{\begin{array}{ll} \dot{z} & = A_0 z + [F(\delta), 0]\,\bar{y} + \bar{K}_1(\delta)\dot{\bar{y}} + \bar{K}_2(\delta)\bar{y} \\ \bar{y} & = \begin{bmatrix} C_0 \\ 0 \end{bmatrix} z + \bar{\Gamma}_1(\delta)\dot{\bar{y}} + \bar{\Gamma}_2(\delta)\bar{y} \end{array}\right. \tag{23}$$

where $\bar{\Gamma}_1(\delta)$, $\bar{\Gamma}_2(\delta)$, A_0, C_0 and $F(\delta)$ are defined in (15), (18) and (20) respectively, with

$$\begin{array}{l} \bar{K}_1(\delta) = T(\delta)K_1(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{n_z \times p}[\delta] \\ \bar{K}_2(\delta) = T(\delta)K_2(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{n_z \times p}[\delta] \end{array} \tag{24}$$

where $n_z = rl^*$.

UI observer

Theorem

If Assumption 1 and Assumption 2 are both satisfied for system (2), then the following dynamics :

$$\left\{\begin{array}{ll} \dot{\xi} & = L_0\xi + J(\delta)\Lambda(\delta)y \\ \hat{z} & = \xi + H(\delta)\Lambda(\delta)y \\ \hat{x} & = T_L^{-1}(\delta)\hat{z} \end{array}\right. \tag{25}$$

with $T_L^{-1}(\delta)$ *being defined in (19), and*

$$\begin{array}{ll} L_0 & = A_0 - G_0C_0 \\ H(\delta) & = \bar{K}_1(\delta) - [G_0, 0]\,\bar{\Gamma}_1(\delta) \\ J(\delta) & = [F(\delta), 0] + \bar{K}_2(\delta) + L_0H(\delta) - [G_0, 0]\,\bar{\Gamma}_2(\delta) + [G_0, 0] \end{array} \tag{26}$$

where G_0 *is a constant matrix which makes* $(A_0 - G_0C_0)$ *Hurwitz, is an exponential unknown input observer for system (2).*

Proof

Denote $e_z = z - \hat{z}$, $e_x = x - \hat{x}$, then we have

$$\begin{aligned} \dot{e}_z &= \dot{z} - \dot{\xi} - H(\delta)\bar{\bar{y}} \\ &= A_0 z - L_0\hat{z} + \left[\bar{F}(\delta) + \bar{K}_2(\delta) + L_0 H(\delta) - J(\delta)\right]\bar{y} + \left[\bar{K}_1(\delta) - H(\delta)\right]\bar{\bar{y}} \end{aligned}$$

Since(A_0, C_0) is observable, then $\exists G_0$ such that $(A_0 - G_0 C_0)$ is Hurwitz. With the chosen matrix G_0, determine L_0, $H(\delta)$ and $J(\delta)$ defined in (26), then

$$\begin{aligned} \dot{e}_z &= A_0 z - L_0\hat{z} + \left[\bar{G}_0\bar{\Gamma}_2(\delta) - \bar{G}_0\right]\bar{y} + \bar{G}_0\bar{\Gamma}_1(\delta)\bar{\bar{y}} \\ &= \left[A_0 - G_0 C_0\right] e_z \end{aligned}$$

Since $x = T_L^{-1}(\delta)z$, then $e_x = x - \hat{x} = T_L^{-1}(\delta)e_z$ is governed by :

$$\dot{e}_x = T_L^{-1}(\delta)\left(A_0 - G_0 C_0\right) T(\delta) e_x$$

Procedure

- Compute the unimodular matrix $U(\delta)$ over $\mathbb{R}[\delta]$ which transforms the following matrix into its Hermite form :

$$U(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} V(\delta) \\ 0 \end{bmatrix}$$

with $V(\delta)$ being full row rank over $\mathbb{R}[\delta]$, and calculate $\bar{V}(\delta)$ such that $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$. Then we obtain the gain matrix $W(\delta) = [\bar{V}(\delta), 0]U(\delta)$;

Procedure

- With the obtained matrix $W(\delta)$, decompose it as $W(\delta) = \begin{pmatrix} K(\delta) \\ \Gamma(\delta) \end{pmatrix} = \begin{bmatrix} K_1(\delta) & K_2(\delta) \\ \Gamma_1(\delta) & \Gamma_2(\delta) \end{bmatrix}$, then transform system (2) into (14) with $\bar{A}(\delta) = A(\delta) - K(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix}$, $\tilde{C}(\delta) = C(\delta) - \Gamma\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix}$, and find the unimodular matrix $\Lambda(\delta)$ over $\mathbb{R}[\delta]$ such that $\Lambda(\delta)\tilde{C}(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix}$;

Procedure

- After having obtained $\bar{A}(\delta)$ and $\bar{C}(\delta)$, deduce $T(\delta)$ defined in (19) and $F(\delta)$ defined in (20) ;
- Deduce A_0 and C_0 defined in (18), $\bar{\Gamma}_1(\delta)$ and $\bar{\Gamma}_2(\delta)$ defined in (15), $\bar{K}_1(\delta)$ and $\bar{K}_2(\delta)$ defined in (24) ;
- Design the observer of the form (25) by choosing the matrices L_0, $H(\delta)$ and $J(\delta)$ defined in (26).

Example

Consider the following example :

$$A(\delta) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & \delta & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \delta & 1 & 1 & -1 \end{bmatrix}, B(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & \delta \end{bmatrix}$$

$$C(\delta) = \begin{bmatrix} \delta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, D(\delta) = \begin{pmatrix} 1 & \delta \\ 0 & 0 \\ 1 & \delta \end{pmatrix}$$

Verification of assumptions

There exist $k^* = 3$ such that $M_{k^*} = M_{k^*+1} = \mathrm{I}_4$, thus Assumption 1 is satisfied. Moreover, by calculating the invariant factors we have

$$\begin{aligned}
\mathrm{Inv}_{\mathrm{S}} \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} &= \mathrm{Inv}_{\mathrm{S}} \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} \\
&= \{1, 1, 1\}
\end{aligned}$$

therefore Assumption 2 is satisfied as well. According to the theorem, there exists a Luenberger-like observer.

Step 1

We can find

$$U(\delta)=\begin{bmatrix} \delta & 0 & -\delta & 1 & 0 & \delta \\ -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, V(\delta)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \delta \end{bmatrix}$$

and

$$\bar{V}(\delta)=\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \delta & \delta & 0 & \delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

such that $\bar{V}(\delta)V(\delta)=\begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$

Step 1

This gives us

$$W(\delta) = \begin{bmatrix} \dfrac{K_1(\delta)}{\Gamma_1(\delta)} & \dfrac{K_2(\delta)}{\Gamma_2(\delta)} \end{bmatrix} = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \delta-1 & 0 & 1-\delta & 1 & 0 & \delta-1 \\ -\delta & 0 & \delta & 0 & 0 & -\delta \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Step 2

With the above deduced $W(\delta)$, we obtain :

$$\bar{A}(\delta) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & \delta & 0 & 0 \\ \delta^2-\delta & \delta^2-1 & -\delta^2+2\delta & -2\delta+2 \\ -\delta^2 & 1-\delta-\delta^2 & 1-\delta+\delta^2 & 2\delta-1 \end{bmatrix}$$

and $\tilde{C}(\delta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$, thus we can choose $\Lambda(\delta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$,
which gives $\bar{C}(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$.

Then we have $\bar{\mathscr{O}}_3(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -1 & \delta & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -\delta & 1+\delta^2 & -1 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$,

$\left[\bar{\mathscr{O}}_3(\delta)\right]_L^{-1} = \begin{bmatrix} \delta & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \delta & 0 & -1 & 0 \\ \delta^2-1 & 1 & -2\delta & 0 & 1 & 0 \end{bmatrix}$, which gives

$$F(\delta) = \begin{bmatrix} F_1(\delta) \\ \hdashline F_2(\delta) \\ \hdashline F_3(\delta) \end{bmatrix} = \begin{bmatrix} -2-\delta^2+5\delta & 0 \\ 0 & 0 \\ \hdashline 1+3\delta-5\delta^2+\delta^3 & 0 \\ 0 & 0 \\ \hdashline 3-4\delta-\delta^2+\delta^3 & 2\delta-2 \\ 0 & 1 \end{bmatrix} \tag{27}$$

Step 3

Then we obtain the bicausal generalized change of coordinates $z = T(\delta)x$ where

$$T(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -1 & 2-4\delta+\delta^2 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -2+4\delta-\delta^2 & -\delta+\delta^2 & -1 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$$

with

$$[T(\delta)]_L^{-1} = \begin{bmatrix} 2-4\delta+\delta^2 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -19\delta^2+15\delta+8\delta^3-4-\delta^4 & 0 & -4\delta+2+\delta^2 & 0 & -1 & 0 \\ 15\delta^2-13\delta-7\delta^3+4+\delta^4 & 0 & 3\delta-2-\delta^2 & 1 & 1 & 0 \end{bmatrix}$$

Step 4

With the deduced change of coordinates, we have

$$A_0 = \left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], C_0 = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc} \bar{K}_1(\delta) & \bar{K}_2(\delta) \end{array}\right] = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & \delta - 1 & 1 - \delta & 0 & 1 - \delta & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc} \bar{\Gamma}_1(\delta) & \bar{\Gamma}_2(\delta) \end{array}\right] = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Step 4

For the simulation setting, we can choose

$$G_0 = \begin{bmatrix} 85 & 0 & 2000 & 0 & 12500 & 0 \\ 0 & 85 & 0 & 2000 & 0 & 12500 \end{bmatrix}^T$$

such that $(A_0 - G_0C_0)$ has negative eigenvalues $(-10, -10, -25, -25, -50, -50)$.

Step 4

And finally we obtain the following gain matrices :

$$L_0 = \begin{bmatrix} -85 & 0 & 1 & 0 & 0 & 0 \\ 0 & -85 & 0 & 1 & 0 & 0 \\ -2000 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2000 & 0 & 0 & 0 & 1 \\ -12500 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12500 & 0 & 0 & 0 & 0 \end{bmatrix}, H(\delta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & \delta-1 & 1-\delta \\ 0 & 1 & -1 \end{bmatrix}$$

$$J(\delta) = \begin{bmatrix} 83+5\delta-\delta^2 & 0 & 0 \\ 0 & 0 & 0 \\ 2001-5\delta^2+3\delta+\delta^3 & \delta-1 & 1-\delta \\ 0 & 0 & 0 \\ 12503-4\delta-\delta^2+\delta^3 & \delta-1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Simulation

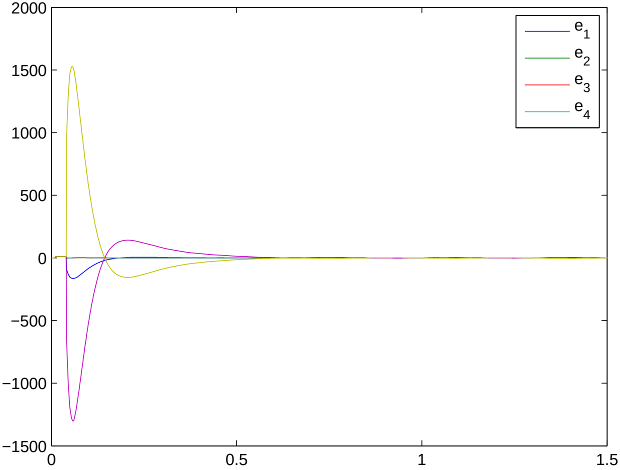

FIGURE : The observation error for $h = 0.01s$.

Simulation

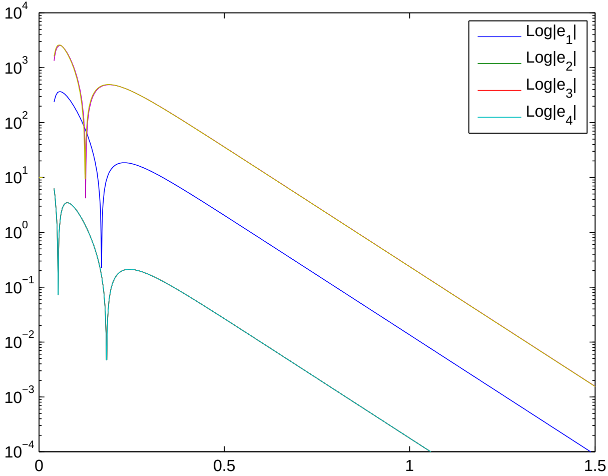

FIGURE : The observation error (in log scale) for $h = 0.01s$.

Simulation

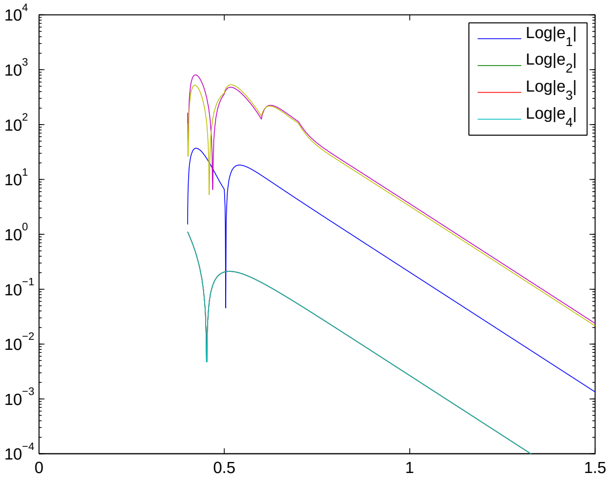

FIGURE : The observation error (in log scale) for $h = 0.1s$.

Conclusions

- The studied system
  - The class of linear time-delay systems is quite larger than that those in the literature since we consider unknown inputs in both the state equation and in the system output.
  - Moreover, commensurate delays are allowed to appear in the state, input, and in the output also.
- Observability analysis
  - We have proposed to tackle the observability of linear commensurable time delay systems with UI using three different definitions ;
  - We have given sufficient conditions allowing for the system to be UIO, BUIO, or FUIO, respectively.
- Observer design
  - We have matched the BUIO with the observability condition required in Hou et al. (2002) for the observer design of linear time-delay systems without inputs.
  - The required conditions for the observer design are considerably relaxed in the sense that they coincide with the NS conditions for the unknown input observer design of LTI.

Thanks !

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