

Observability and observer design for linear TDS

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- Motivation
- Part I : Observability analysis
- Part II : Observer design
- Simulations
- Conclusions

Motivation

- Goal : Using available measurements to estimate the internal states ;
- Reason : Controller needs sometimes full/partial states, or sometimes we need to filter the noisy measurement
- This topic has been widely studied in the literature for different types of systems :
 - Linear system (Kalman et al. (1969); Silverman (1969); Molinari (1976); Bhattacharyya (1978); Hautus (1983); Darouach et al. (1994); Yang and Wilde (1988); Hou and Muller (1992); Kudva et al. (1980); Wang et al. (1975); Hostetter and Meditch (1973)...),
 - Nonlinear system (Hermann and Krener (1977); Hirschorn (1979); Krener (1985); Respondek (1990); Diop and Fliess (1991); Isidori (1995); Gauthier et al. (1992).; Boutat et al. (2001); Barbot et al. (2005); Hammouri et al. (1994); Farza (2005)...),
 - Time-delay (Bhat and Koivo (1976); Fattouh et al. (1999a); Germani et al. (1998); Hou et al. (2002); Conte et al. (2003); Fliess and Mounier (1998); Sename (2001); Fu et al. (2004); Darouach (2006a)...)
 - ...

Observability VS Observer

Given a general nonlinear system :

- First question : It is possible to estimate full/partial state ? ([Analyze Observability/Detectability](#))
 - depending on the known information (output, input, parameters,...)
 - depending as well on the system (with/without delay, singular,...)
- Second question : If yes, then how to estimate them ? ([Design Differentiator/Observer](#))
 - Differentiator : Algebraic, HOSM, Homogeneity, High-gain,...
 - Observer : Luenberger, Kalman, High-gain, adaptive, sliding mode,...

They are not equivalent !

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For a general linear time-delay systems :

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{k_a} A_i x(t - \tau_i) + \sum_{i=0}^{k_b} B_i u(t - \tau_i) \\ y(t) &= \sum_{i=0}^{k_c} C_i x(t - \tau_i) + \sum_{i=0}^{k_d} D_i u(t - \tau_i)\end{aligned}$$

we are going to answer :

- under which condition the system is observable (**backward and forward**) ;
- under which condition the state can be estimate via a simple **Luenberger-like** observer ;

Part I : Observability analysis

Some relevant existing works

- Known input :
 - one delay + only in the state (Fattouh et al. (1999b); Darouach (2001)...))
 - multiple delays both in the state and output (Emre and Khargonekar (1982); Sename (1997); Hou and Muller (1992)...))
- Unknown input :
 - one delay + only in the state (Conte et al. (2003); Sename (2001); Darouach (2006b)...))
 - multiple delays both in the state and output (Not yet solved!)

Assumption

Considered system :

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{k_a} A_i x(t - \tau_i) + \sum_{i=0}^{k_b} B_i u(t - \tau_i) \\ y(t) &= \sum_{i=0}^{k_c} C_i x(t - \tau_i) + \sum_{i=0}^{k_d} D_i u(t - \tau_i)\end{aligned}$$

- Assumptions :

- delay is commensurate delay, i.e. $\exists h$ s.t. $\tau_i = ih$;
- input is unknown ;

The studied can be rewritten as :

$$\begin{cases} \dot{x}(t) = \sum_{i=0}^{k_a} A_i x(t - ih) + \sum_{i=0}^{k_b} B_i u(t - ih) \\ y(t) = \sum_{i=0}^{k_c} C_i x(t - ih) + \sum_{i=0}^{k_e} D_i u(t - ih) \end{cases} \quad (1)$$

Delay operator

Delay operator $\delta : x(t) \rightarrow x(t-h)$ with $\delta^k x(t) = x(t-kh)$, $k \in \mathbb{N}_0$;

$\mathbb{R}[\delta]$: the polynomial ring of δ over the field \mathbb{R} ;

i.e. for any $a(\delta) = \sum_{i=0}^{d_a} a_i \delta^i \in \mathbb{R}[\delta]$ with $a_i \in \mathbb{R}$:

$$\begin{aligned} a(\delta) + b(\delta) &= \sum_{i=0}^{\max\{d_a, d_b\}} (a_i + b_i) \delta^i \\ a(\delta)b(\delta) &= \sum_{i=0}^{d_a} \sum_{j=0}^{d_b} a_i b_j \delta^{i+j} \end{aligned}$$

it means

$$a(\delta) + b(\delta) = b(\delta) + a(\delta)$$

and

$$a(\delta)b(\delta) = b(\delta)a(\delta)$$

$\mathbb{R}[\delta]$ is a commutative ring!

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Consider the system

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then we have :

$$\begin{cases} \dot{x}(t) &= A(\delta)x(t) + B(\delta)u(t) \\ y(t) &= C(\delta)x(t) + D(\delta)u(t) \end{cases} \quad (2)$$

with $A(\delta) := \sum_{i=0}^{k_a} A_i \delta^i$, $B(\delta) := \sum_{i=0}^{k_b} B_i \delta^i$, $C(\delta) := \sum_{i=0}^{k_c} C_i \delta^i$, $D(\delta) := \sum_{i=0}^{k_d} D_i \delta^i$.

$A(\delta)$, $B(\delta)$, $C(\delta)$, and $D(\delta)$ are matrices over the polynomial ring $\mathbb{R}[\delta]$!

What is the interest to introduce δ ?

LTI system

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

with A , B , C and D being matrices over \mathbb{R} .

LTD system

$$\begin{cases} \dot{x} &= A(\delta)x + B(\delta)u \\ y &= C(\delta)x + D(\delta)u \end{cases}$$

with $A(\delta)$, $B(\delta)$, $C(\delta)$, and $D(\delta)$ being matrices over $\mathbb{R}[\delta]$.

- Interest : The same structure of the systems implies the existing results (Observability/Observer) for LTI **MIGHT** be applicable for LTD!
- Attention : The different definition of the matrices implies particular attentions **SHOULD** be paid when adapting those results!

Dependence of delay

Since TDS contains delay, the observability depends as well on delay (*backward or forward*) !

Definition

[BUIO] System (1) is **backward UIO (BUIO)** on $[t_1, t_2]$ if for each $\tau \in [t_1, t_2]$ there exist $t'_1 < t'_2 \leq \tau$ such that, for all u and φ ,

$$y(t; \varphi, u) = 0 \text{ for all } t \in [t'_1, t'_2] \text{ implies } x(\tau; \varphi, u) = 0.$$

Definition

[FUIO] System (1) is **forward UIO (FUIO)** on $[t_1, t_2]$ if for each $\tau \in [t_1, t_2]$ there exist $t'_2 > t'_1 \geq \tau$ such that, for all u and φ ,

$$y(t; \varphi, u) = 0 \text{ for all } t \in [t'_1, t'_2] \text{ implies } x(\tau; \varphi, u) = 0.$$

- These definitions are essentially formulated following the observability definitions given in [Kalman et al. \(1969\)](#) and the strong observability given in [Hautus \(1983\)](#) for LTI ;
- UIO : reconstruction using past, actual, and future values of y ;
- FUIO : reconstruction using actual and future values of y ;
- **BUIO : reconstruction using past and actual values of y ;**
- Either BUIO or FUIO implies UIO ;
- BUIO and FUIO do not exclude each other.

Ex :

$$\dot{x}_1 = x_2 \quad 6.797011 \quad T_f \quad 4.40 \quad 10.9091 \quad T_f \quad 4.559.9199 \quad 0 \quad 978. \quad c \quad 260.0 \quad 10 \quad 0 \quad 0 \quad \text{cm BT 5238.38}$$

Linear TI system

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

NS condition of strong observability ([Hautus \(1983\)](#)[Trentelman et al. \(2001\)](#)) :

$$\text{rank} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = n + \text{rank} \begin{bmatrix} B \\ D \end{bmatrix} \text{ for all } s \in \mathbb{C} \quad (3)$$

Easily to check, but sometimes we need more (ex : the relation)!

LTI system

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

Iterative algorithm ([Molinari \(1976\)](#); [Silverman \(1969\)](#)) :

$$\begin{aligned} \Delta_0 &\triangleq 0, \quad G_0 \triangleq C, \quad F_0(\delta) \triangleq D, \quad N_0 = 0 \\ \begin{bmatrix} F_{k+1} & G_{k+1} \\ 0 & \Delta_{k+1} \end{bmatrix} &\triangleq T_k \begin{bmatrix} \Delta_k B & \Delta_k A \\ F_k & G_k \end{bmatrix} \\ N_{k+1} &\triangleq \begin{bmatrix} N_k \\ \Delta_{k+1} \end{bmatrix} \end{aligned}$$

then system is SO iff $\exists k \in \mathbb{N}_0$, s.t. $\text{rank} N_{k+1} = \text{rank} N_k = n$.

Difficulties from LTI to LTDS (Inverse problem) :

- over \mathbb{R} , $\text{rank}N_k = n$ implies the existence of the left inverse of N_k ;
- over $\mathbb{R}[\delta]$, it is not trivial ;

ex :

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \text{ VS } \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & \delta \end{bmatrix} \text{ VS } \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & \delta \end{bmatrix}$$

Definition



Properties : Hermite form

For $P(\delta)$ be a matrix of $q \times s$ dimension with rank equal to r (clearly $r \leq \min\{q, s\}$), there exists an invertible matrix $T(\delta)$ over $\mathbb{R}[\delta]$ such that $P(\delta)$ is put into (column) Hermite form :

$$T(\delta)P(\delta) = \begin{bmatrix} P_1(\delta) \\ 0 \end{bmatrix}$$

where $P_1(\delta)$ is of $r \times s$ dimension, and $\text{rank}P_1 = r$.

Properties : Smith form

For $P(\delta)$ be a matrix of $q \times s$ dimension with rank equal to r , there exist two invertible matrices $U(\delta)$ and $V(\delta)$ over $\mathbb{R}[\delta]$ such that $P(\delta)$ is reduced to its Smith form :

$$U(\delta)P(\delta)V(\delta) = \begin{bmatrix} \text{diag}(\psi_1(\delta) \cdots \psi_r(\delta)) & 0 \\ 0 & 0 \end{bmatrix}$$

where $\{\psi_i(\delta)\}$'s are called **invariant factors**, noted as :

$$\text{Inv}_S[P(\delta)] = \{\psi_i(\delta)\}_{1 \leq i \leq r}$$

Lemma

A polynomial matrix $D(\delta) \in \mathbb{R}^{p \times m}[\delta]$ is left (or right) unimodular (or invertible) over $\mathbb{R}[\delta]$ if and only if the following conditions are satisfied :

- 1 $\text{rank}_{\mathbb{R}[\delta]} D(\delta) = m \leq p$ (or $\text{rank}_{\mathbb{R}[\delta]} D(\delta) = p \leq m$) ;
- 2 $\text{Inv}_S [D(\delta)] \subset \mathbb{R}$.

It is said to be unimodular (or invertible) over $\mathbb{R}[\delta]$ if and only if the following conditions are satisfied :

- 1 $\text{rank}_{\mathbb{R}[\delta]} D(\delta) = p = m$;
- 2 $\text{Inv}_S [D(\delta)] \subset \mathbb{R}$.

Similar algorithm for LTDS

Following the ideas of [Silverman \(1969\)](#) and [Molinari \(1976\)](#), define $\{\Delta_k(\delta)\}$ matrices generated by the following algorithm :

$$\begin{aligned} \Delta_0 &\triangleq 0, \quad G_0(\delta) \triangleq C(\delta), \quad F_0(\delta) \triangleq D(\delta) \\ \begin{bmatrix} F_{k+1}(\delta) & G_{k+1}(\delta) \\ 0 & \Delta_{k+1}(\delta) \end{bmatrix} &\triangleq T_k(\delta) \begin{bmatrix} \Delta_k(\delta)B(\delta) & \Delta_k(\delta)A(\delta) \\ F_k(\delta) & G_k(\delta) \end{bmatrix} \end{aligned} \quad (4)$$

and

$$\begin{aligned} M_0(\delta) &\triangleq N_0(\delta) \triangleq \Delta_0, \quad N_{k+1}(\delta) \triangleq \begin{bmatrix} N_k(\delta) \\ \Delta_{k+1}(\delta) \end{bmatrix}, \quad \text{for } k \geq 0 \\ \begin{bmatrix} M_{k+1}(\delta) \\ 0 \end{bmatrix} &\triangleq \begin{bmatrix} \text{diag}(\psi_1^{k+1}(\delta), \dots, \psi_{r_{k+1}}^{k+1}(\delta)) & 0 \\ 0 & 0 \end{bmatrix} \\ &= U_{k+1}(\delta)N_{k+1}(\delta)V_{k+1}(\delta) \end{aligned} \quad (5)$$

where $M_{k+1}(\delta)$ is the Smith form of $N_{k+1}(\delta)$.

Theorem

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Theorem

After a finite number of steps, let's say k^ , the algorithm (4)-(5) converges, i.e., there exists a least integer k^* such that $M_{k^*+1}(\delta) = M_{k^*}(\delta)$. Furthermore, k^* is independent of the choice of $\{T_k\}$, $\{U_k\}$, and $\{V_k\}$ matrices used in (4)-(5).*

Remark

Theorem 8 gives a way to have an upper estimation of how many steps will pass before the rank of M_k does change, if it does.

Corollary

Let k^ be the least integer k such that $M_{k+1} = M_k$, then for all $i \geq 0$, $M_{k^*+i} = M_{k^*}$.*

What is the role of M_{k^*} ?

LTD system

$$\begin{cases} \dot{x} &= A(\delta)x + B(\delta)u \\ y &= C(\delta)x + D(\delta)u \end{cases}$$

Following the algorithm :

$$\begin{bmatrix} F_{k+1}(\delta) & G_{k+1}(\delta) \\ 0 & \Delta_{k+1}(\delta) \end{bmatrix} \triangleq T_k(\delta) \begin{bmatrix} \Delta_k(\delta)B(\delta) & \Delta_k(\delta)A(\delta) \\ F_k(\delta) & G_k(\delta) \end{bmatrix}$$

finally $\exists t^*$ such that

$$N_{k^*}(\delta)x(t) = Y(t) \text{ for all } t \geq t^* \quad (6)$$

where $Y(t)$ is the vector containing only the derivatives of the y and its delay, and

$$M_{k^*}(\delta) = U_{k^*}(\delta)N_{k^*}(\delta)V_{k^*}(\delta)$$

where $M_{k^*}(\delta)$ is the Smith form of $N_{k^*}(\delta)$.

Theorem

System is UIO, if $M_{k^*}(\delta)$ has n invariant factors of the form $a\delta^j$, where $a \in \mathbb{R}_{\neq 0}$, $j \in \mathbb{N}_0$. The formula to reconstruct $x(t)$ is

$$x(t) = \begin{bmatrix} V_{k^*} M_{k^*}^{-1} & 0 \end{bmatrix} U_{k^*} Y(t), \quad t > t_1^* \quad (7)$$

Furthermore, the i -th entry of $x(t)$ is given by an expression of the form :

$$x_i(t) = \sum_{k,j} q_{k,j} y_k^{(j)}(t) \quad (8)$$

where $y_k^{(i)}(t)$ is the i -th derivative of the k -th entry of $y(t)$ and $0 \neq q_{j,k}^i \in \mathbb{R}[\delta, \delta^{-1}]$.

It is obvious since $Y(t) = N_{k^*}(\delta)x(t)$ (for all $t \geq t_1^*$), thus $\exists t_1^*$ such that $x(t) = \begin{bmatrix} V_{k^*} M_{k^*}^{-1} & 0 \end{bmatrix} U_{k^*} Y(t), \quad t > t_1^* \dots$

Theorem

System is BUIO, if system is UIO, and one of the following equivalent conditions are satisfied :

- 1 $q_{j,i}$ of (8) belongs to $\mathbb{R}[\delta]$;
- 2 $M_{k^*}(\delta)$ has n invariant factors belonging to \mathbb{R} ;

The first condition is evident since $x_i(t) = \sum_{k,j} q_{k,j} y_k^{(j)}(t)$, and the second is due to the fact that $x(t) = \begin{bmatrix} V_{k^*} M_{k^*}^{-1} & 0 \end{bmatrix} U_{k^*} Y(t)$.

Remark

For the case $h = 0$, the condition that M_{k^*} is invertible over \mathbb{R} is also NSC for the system to be strong observability ([Molinari \(1976\)](#))

Theorem

System is BUIO, if system is UIO, and every polynomial $q_{j,i}$ of (8) belongs to $\mathbb{R}[\delta^{-1}]$.

Example 1

After applying the algorithm, $M_3 = M_4 = \text{diag}(1, 1, \delta, \delta, \delta^2)$. Thus, in this case $k^* = 3$, and the system is UIO. Explicitly, we have that the state vector can be expressed as

$$x_1 = y_1$$

$$x_2 = -\delta^{-1}y_1 + \delta^{-1}\dot{y}_1 + \frac{1}{2}(1 - \delta^{-1})\dot{y}_2 - \frac{1}{2}\delta^{-1}\dot{y}_3$$

$$x_3 = \delta^{-1}y_1 + y_2 - \delta^{-1}\dot{y}_1 - \frac{1}{2}(1 - \delta^{-1})\dot{y}_2 + \frac{1}{2}\delta^{-1}\dot{y}_3 \\ - \frac{1}{2}(1 - \delta^{-1})\ddot{y}_2 + \frac{1}{2}\delta^{-1}\ddot{y}_3$$

$$x_4 = \frac{1}{2}(1 - \delta^{-1})\dot{y}_2 - \frac{1}{2}\delta^{-1}\dot{y}_3$$

$$x_5 = (\delta^{-2} - \delta^{-1})y_1 + (\delta^{-1} - 1)y_2 + \delta^{-1}y_3 \\ + (\delta^{-1} - \delta^{-2})\dot{y}_1 + \frac{1}{2}(1 - \delta^{-1})^2\dot{y}_2 + \frac{1}{2}(\delta^{-2} - \delta^{-1})\dot{y}_3$$

Therefore, the system **is FUIO**.

Example 2

let us consider

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & \delta & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \delta \end{pmatrix}$$
$$C = \begin{pmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -2\delta & 0 \\ 0 & 0 \end{pmatrix}$$

In this example, we have $M_2 = M_3 = \text{diag}(1, 1, 1)$. It is UIO, and **BUIO**. Indeed, it is easy to verify that the state variables can be expressed as

$$\begin{aligned} x_1(t) &= -\dot{y}_2(t) + \delta y_2(t), & x_2(t) &= y_2(t) \\ x_3(t) &= -\ddot{y}_2(t) + \delta \dot{y}_2(t) + y_2(t) \end{aligned}$$

Part II : Observer design

Problem statement

Consider LTDS :

$$\begin{cases} \dot{x} &= A(\delta)x + B(\delta)u \\ y &= C(\delta)x + D(\delta)u \end{cases}$$

where $x(t) \in \mathbb{R}^{n_x}$, $y(t) \in \mathbb{R}^p$, and $u(t) \in \mathbb{R}^m$.

Assumption

For the studied system, there exists a least integer $k^ \in \mathbb{N}_0$ such that $\text{rank}_{\mathbb{R}[\delta]} M_{k^*}(\delta) = n_x$, and $M_{k^*}(\delta)$ is unimodular over $\mathbb{R}[\delta]$.*

The above assumption implies that the system is BUIO. It means :

- 1 the iterative algorithm yields $x_i(t) = \sum_{k,j} q_{k,j} y_k^{(j)}(t)$;
- 2 we can use the differentiator to estimate the state.

Question : it is possible to use the basic observer ?

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Consider LTI system :

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

there exists a Luenberger-like observer ([Trentelman et al. \(2001\)](#)) :

$$\begin{aligned} \dot{\xi} &= P\xi + Qy \\ \hat{x} &= \xi + Ky \end{aligned}$$

if and only if

$$\text{rank} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = n + \text{rank} \begin{bmatrix} B \\ D \end{bmatrix} \text{ for all } s \in \mathbb{C} \quad (9)$$

and

$$\text{rank} \begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} B \\ D \end{bmatrix} + \text{rank} D \quad (10)$$

Need another condition

Remark

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When treating linear systems without delay, the conditions imposed in Assumption 1 is equivalent to :

$$\text{rank} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = n + \text{rank} \begin{bmatrix} B \\ D \end{bmatrix} \text{ for all } s \in \mathbb{C} \quad (11)$$

We need to find another condition which should be equivalent to the following one (for the case $h = 0$) :

$$\text{rank} \begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} B \\ D \end{bmatrix} + \text{rank} D$$

Proposed the second condition

Assumption

It is assumed that

$$\text{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} = \text{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} \quad (12)$$

Remark

When treating linear systems without delay, the condition imposed in Assumption 2 is equivalent to :

$$\text{rank} \begin{bmatrix} CB & D \\ D & 0 \\ B & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} B \\ D \end{bmatrix} + \text{rank} D$$

Lemma

Suppose Assumption 2 is satisfied, then there exists a matrix $W(\delta)$ satisfying the following conditions :

- ① $W(\delta) \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$;
- ② for any matrix $J(\delta)$ such that $J(\delta) \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} = 0$, then $J(\delta)W(\delta) = 0$.

The first condition is obvious since

$$\text{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} = \text{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix}.$$

Transform into Hermite form :

$$U(\delta) \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} V(\delta) \\ 0 \end{bmatrix}$$

Since $\exists W(\delta)$ s.t. $W(\delta) \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$, then $\exists \bar{V}(\delta)$

s.t. $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$. Finally select $W(\delta) = [\bar{V}(\delta), 0]U(\delta)$.

As $V(\delta)$ is full row rank over $\mathbb{R}[\delta]$, then for any matrix $J(\delta)$ such that

$J(\delta) \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} = 0$ we always have $J(\delta)\bar{V}(\delta) = 0$, which implies

$J(\delta)W(\delta) = 0$.

For LTDS

$$\begin{cases} \dot{x} &= A(\delta)x + B(\delta)u \\ y &= C(\delta)x + D(\delta)u \end{cases}$$

decompose $W(\delta) = [W_1(\delta), W_2(\delta)]$, then we have $W_1(\delta)D(\delta) = 0$. Thus, we obtain

$$W_1(\delta)y = W_1(\delta)C(\delta)x$$

which yields

$$W_1(\delta)\dot{y} = W_1(\delta)C(\delta)A(\delta)x + W_1(\delta)C(\delta)B(\delta)u$$

System decomposition

Since $W(\delta) \begin{bmatrix} C(\delta)B(\delta) \\ D(\delta) \end{bmatrix} = \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix}$, then we have

$$\begin{aligned} W_1(\delta)\dot{y} + W_2(\delta)y &= W(\delta) \begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} x + W(\delta) \begin{bmatrix} C(\delta)B(\delta) \\ D(\delta) \end{bmatrix} u \\ &= W(\delta) \begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} x + \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u \end{aligned}$$

Thus $\begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u = W(\delta) \begin{bmatrix} \dot{y} - C(\delta)A(\delta) \\ y - C(\delta) \end{bmatrix}$

System decomposition

Decompose again the matrix $W(\delta)$ as

$$W(\delta) = [W_1(\delta), W_2(\delta)] = \begin{pmatrix} K(\delta) \\ \Gamma(\delta) \end{pmatrix} = \begin{bmatrix} K_1(\delta) & K_2(\delta) \\ \Gamma_1(\delta) & \Gamma_2(\delta) \end{bmatrix}$$

since

$$\begin{aligned} \dot{x} &= A(\delta)x + B(\delta)u \\ y &= C(\delta)x + D(\delta)u \end{aligned}, \quad \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u = W(\delta) \begin{bmatrix} \dot{y} - C(\delta)A(\delta)x \\ y - C(\delta)x \end{bmatrix}$$

then system (2) becomes

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x + K_1(\delta)\dot{y} + K_2(\delta)y \\ y &= \tilde{C}(\delta)x + \Gamma_1(\delta)\dot{y} + \Gamma_2(\delta)y \end{aligned} \quad (13)$$

where $\bar{A}(\delta) = A - K \begin{bmatrix} CA \\ C \end{bmatrix}$ and $\tilde{C}(\delta) = C - \Gamma \begin{bmatrix} CA \\ C \end{bmatrix}$.

System decomposition

Suppose $\text{rank}_{\mathbb{R}[\delta]} \tilde{C}(\delta) = r \leq p$, then $\exists \Lambda(\delta)$ s.t. $\Lambda(\delta)\tilde{C}(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix}$. By noting $\bar{y} = \Lambda(\delta)y$, finally system (13) can be decomposed into :

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x + K_1(\delta)\Lambda^{-1}(\delta)\dot{\bar{y}} + K_2(\delta)\Lambda^{-1}(\delta)\bar{y} \\ \bar{y} &= \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix} x + \bar{\Gamma}_1(\delta)\dot{\bar{y}} + \bar{\Gamma}_2(\delta)\bar{y} \end{aligned} \quad (14)$$

where

$$\begin{aligned} \bar{\Gamma}_1(\delta) &= \Lambda(\delta)\Gamma_1(\delta)\Lambda^{-1}(\delta) \\ \bar{\Gamma}_2(\delta) &= \Lambda(\delta)\Gamma_2(\delta)\Lambda^{-1}(\delta) \end{aligned} \quad (15)$$

Define the following polynomial matrix over $\mathbb{R}[\delta]$:

$$\bar{\mathcal{O}}_l(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ \bar{C}(\delta)\bar{A}(\delta) \\ \vdots \\ \bar{C}(\delta)\bar{A}^{l-1}(\delta) \end{bmatrix} \in \mathbb{R}^{rl \times n_x}[\delta] \quad (16)$$

where $l \in \mathbb{N}_0$, and let us recall a useful result stated in [Hou and Muller \(1992\)](#).

Theorem of Hou and Muller (1992)

There exists a bicausal generalized change of coordinates $z = T(\delta)x$ which transforms the following system :

$$\begin{aligned}\dot{x} &= \bar{A}(\delta)x \\ \bar{y} &= \bar{C}(\delta)x\end{aligned}\quad (17)$$

with $\text{rank}_{\mathbb{R}[\delta]} \bar{C}(\delta) = r$ into the following observer normal form :

$$\begin{cases} \dot{z} = A_0 z + F(\delta)\bar{y} \\ \bar{y} = C_0 z \end{cases}$$

where $F(\delta) = [F_1^T(\delta), \dots, F_{l^*}^T(\delta)]^T$ and

$$A_0 = \begin{bmatrix} 0 & \mathbf{I}_r & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{I}_r \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{rl^* \times rl^*}, C_0 = [\mathbf{I}_r, 0, \dots, 0] \in \mathbb{R}^{r \times rl^*} \quad (18)$$

if and only if there exists a least integer $l^* \in \mathbb{N}_0$ such that $\bar{\mathcal{O}}_{l^*}(\delta)$ defined in (16) is left unimodular over $\mathbb{R}[\delta]$.

Theorem of Hou and Muller (1992)

Moreover, the bicausal generalized change of coordinates $z = T(\delta)x$ with $T(\delta) = \text{col}\{T_1(\delta), \dots, T_{l^*}(\delta)\}$ is defined as follows :

$$\begin{cases} T_1(\delta) = \bar{C}(\delta) \\ T_{i+1}(\delta) = T_i(\delta)\bar{A}(\delta) - F_i(\delta)\bar{C}(\delta), \text{ for } 1 \leq i \leq l^* - 1 \end{cases} \quad (19)$$

with $F_i(\delta)$ being determined through the following equations :

$$[F_{l^*}(\delta), \dots, F_1(\delta)] = \bar{C}(\delta)\bar{A}^{l^*}(\delta) [\bar{\theta}_{l^*}(\delta)]_L^{-1} \quad (20)$$

Lemma

If Assumption 1 is satisfied for the quadruple $(A(\delta), B(\delta), C(\delta), D(\delta))$ in system (2), then there exists a least integer $l^* \in \mathbb{N}_0$ such that

$$\mathcal{O}_{l^*}(\delta) = \begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \\ \vdots \\ C(\delta)A^{l^*-1}(\delta) \end{bmatrix} \in \mathbb{R}^{pl^* \times n_x}[\delta] \quad (21)$$

is left unimodular over $\mathbb{R}[\delta]$.

Lemma

If Assumption 1 is satisfied for the quadruple $(A(\delta), B(\delta), C(\delta), D(\delta))$ defined in (2), then for the deduced system (13) there exists a least integer $l^* \in \mathbb{N}_0$ such that

$$\tilde{\mathcal{O}}_{l^*}(\delta) = \begin{bmatrix} \tilde{C}(\delta) \\ \tilde{C}(\delta)\tilde{A}(\delta) \\ \vdots \\ \tilde{C}(\delta)\tilde{A}^{l^*-1}(\delta) \end{bmatrix} \in \mathbb{R}^{pl^* \times n_x}[\delta] \quad (22)$$

is left unimodular over $\mathbb{R}[\delta]$.

Lemma

If there exists a least integer $l^ \in \mathbb{N}_0$ such that $\tilde{\mathcal{O}}_{l^*}(\delta)$ defined in (22) is left unimodular over $\mathbb{R}[\delta]$, then $\bar{\mathcal{O}}_{l^*}(\delta)$ defined in (16) is left unimodular over $\mathbb{R}[\delta]$.*

Theorem

If Assumption 1 and Assumption 2 are both satisfied for system (2), then for the deduced system (14) there exists a least integer $l^ \in \mathbb{N}_0$ such that $\bar{\mathcal{O}}_{l^*}(\delta)$ defined in (16) is left unimodular over $\mathbb{R}[\delta]$.*

Corollary

If Assumption 1 and Assumption 2 are both satisfied for system (2), then there exists a bicausal generalized change of coordinates $z = T(\delta)x$ defined in (19) such that system (14) can be transformed into :

$$\begin{cases} \dot{z} = A_0 z + [F(\delta), 0] \bar{y} + \bar{K}_1(\delta) \dot{\bar{y}} + \bar{K}_2(\delta) \bar{y} \\ \bar{y} = \begin{bmatrix} C_0 \\ 0 \end{bmatrix} z + \bar{\Gamma}_1(\delta) \dot{\bar{y}} + \bar{\Gamma}_2(\delta) \bar{y} \end{cases} \quad (23)$$

where $\bar{\Gamma}_1(\delta)$, $\bar{\Gamma}_2(\delta)$, A_0 , C_0 and $F(\delta)$ are defined in (15), (18) and (20) respectively, with

$$\begin{aligned} \bar{K}_1(\delta) &= T(\delta)K_1(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{n_z \times p}[\delta] \\ \bar{K}_2(\delta) &= T(\delta)K_2(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{n_z \times p}[\delta] \end{aligned} \quad (24)$$

where $n_z = rl^*$.

Theorem

If Assumption 1 and Assumption 2 are both satisfied for system (2), then the following dynamics :

$$\begin{cases} \dot{\xi} &= L_0 \xi + J(\delta) \Lambda(\delta) y \\ \hat{z} &= \xi + H(\delta) \Lambda(\delta) y \\ \hat{x} &= T_L^{-1}(\delta) \hat{z} \end{cases} \quad (25)$$

with $T_L^{-1}(\delta)$ being defined in (19), and

$$\begin{aligned} L_0 &= A_0 - G_0 C_0 \\ H(\delta) &= \bar{K}_1(\delta) - [G_0, 0] \bar{\Gamma}_1(\delta) \\ J(\delta) &= [F(\delta), 0] + \bar{K}_2(\delta) + L_0 H(\delta) - [G_0, 0] \bar{\Gamma}_2(\delta) + [G_0, 0] \end{aligned} \quad (26)$$

where G_0 is a constant matrix which makes $(A_0 - G_0 C_0)$ Hurwitz, is an exponential unknown input observer for system (2).

Denote $e_z = z - \hat{z}$, $e_x = x - \hat{x}$, then we have

$$\begin{aligned} \dot{e}_z &= \dot{z} - \dot{\hat{z}} - H(\delta)\dot{y} \\ &= A_0 z - L_0 \hat{z} + [\bar{F}(\delta) + \bar{K}_2(\delta) + L_0 H(\delta) - J(\delta)] \bar{y} + [\bar{K}_1(\delta) - H(\delta)] \dot{\bar{y}} \end{aligned}$$

Since (A_0, C_0) is observable, then $\exists G_0$ such that $(A_0 - G_0 C_0)$ is Hurwitz. With the chosen matrix G_0 , determine L_0 , $H(\delta)$ and $J(\delta)$ defined in (26), then

$$\begin{aligned} \dot{e}_z &= A_0 z - L_0 \hat{z} + [\bar{G}_0 \bar{\Gamma}_2(\delta) - \bar{G}_0] \bar{y} + \bar{G}_0 \bar{\Gamma}_1(\delta) \dot{\bar{y}} \\ &= [A_0 - G_0 C_0] e_z \end{aligned}$$

Since $x = T_L^{-1}(\delta)z$, then $e_x = x - \hat{x} = T_L^{-1}(\delta)e_z$ is governed by :

$$\dot{e}_x = T_L^{-1}(\delta) (A_0 - G_0 C_0) T(\delta) e_x$$

- Compute the unimodular matrix $U(\delta)$ over $\mathbb{R}[\delta]$ which transforms the following matrix into its Hermite form :

$$U(\delta) \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} V(\delta) \\ 0 \end{bmatrix}$$

with $V(\delta)$ being full row rank over $\mathbb{R}[\delta]$, and calculate $\bar{V}(\delta)$ such that $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$. Then we obtain the gain matrix $W(\delta) = [\bar{V}(\delta), 0]U(\delta)$;

- With the obtained matrix $W(\delta)$, decompose it as

$$W(\delta) = \begin{pmatrix} K(\delta) \\ \Gamma(\delta) \end{pmatrix} = \begin{bmatrix} K_1(\delta) & K_2(\delta) \\ \Gamma_1(\delta) & \Gamma_2(\delta) \end{bmatrix}, \text{ then transform system (2)}$$

into (14) with $\bar{A}(\delta) = A(\delta) - K(\delta) \begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix},$

$\tilde{C}(\delta) = C(\delta) - \Gamma \begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix},$ and find the unimodular matrix $\Lambda(\delta)$

over $\mathbb{R}[\delta]$ such that $\Lambda(\delta)\tilde{C}(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix};$

- After having obtained $\bar{A}(\delta)$ and $\bar{C}(\delta)$, deduce $T(\delta)$ defined in (19) and $F(\delta)$ defined in (20);
- Deduce A_0 and C_0 defined in (18), $\bar{\Gamma}_1(\delta)$ and $\bar{\Gamma}_2(\delta)$ defined in (15), $\bar{K}_1(\delta)$ and $\bar{K}_2(\delta)$ defined in (24);
- Design the observer of the form (25) by choosing the matrices L_0 , $H(\delta)$ and $J(\delta)$ defined in (26).

Example

Consider the following example :

$$A(\delta) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & \delta & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \delta & 1 & 1 & -1 \end{bmatrix}, B(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & \delta \end{bmatrix}$$

$$C(\delta) = \begin{bmatrix} \delta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, D(\delta) = \begin{pmatrix} 1 & \delta \\ 0 & 0 \\ 1 & \delta \end{pmatrix}$$

Verification of assumptions

There exist $k^* = 3$ such that $M_{k^*} = M_{k^*+1} = \mathbf{I}_4$, thus Assumption 1 is satisfied. Moreover, by calculating the invariant factors we have

$$\begin{aligned} \text{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} &= \text{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} \\ &= \{1, 1, 1\} \end{aligned}$$

therefore Assumption 2 is satisfied as well. According to the theorem, there exists a Luenberger-like observer.

Step 1

We can find

$$U(\delta) = \begin{bmatrix} \delta & 0 & -\delta & 1 & 0 & \delta \\ -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, V(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \delta \end{bmatrix}$$

and

$$\bar{V}(\delta) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \delta & \delta & 0 & \delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

such that $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$

Step 1

This gives us

$$W(\delta) = \left[\begin{array}{c} \frac{K_1(\delta)}{\Gamma_1(\delta)} \\ \frac{K_2(\delta)}{\Gamma_2(\delta)} \end{array} \right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \delta - 1 & 0 & 1 - \delta & 1 & 0 & \delta - 1 \\ -\delta & 0 & \delta & 0 & 0 & -\delta \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 2

With the above deduced $W(\delta)$, we obtain :

$$\bar{A}(\delta) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & \delta & 0 & 0 \\ \delta^2 - \delta & \delta^2 - 1 & -\delta^2 + 2\delta & -2\delta + 2 \\ -\delta^2 & 1 - \delta - \delta^2 & 1 - \delta + \delta^2 & 2\delta - 1 \end{bmatrix}$$

and $\tilde{C}(\delta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$, thus we can choose $\Lambda(\delta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$,

which gives $\bar{C}(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$.

Step 3

Then we have $\bar{\mathcal{O}}_3(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -1 & \delta & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -\delta & 1 + \delta^2 & -1 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix},$

$[\bar{\mathcal{O}}_3(\delta)]_L^{-1} = \begin{bmatrix} \delta & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \delta & 0 & -1 & 0 \\ \delta^2 - 1 & 1 & -2\delta & 0 & 1 & 0 \end{bmatrix},$ which gives

$$F(\delta) = \begin{bmatrix} -\frac{F_1(\delta)}{\bar{F}_2(\delta)} \\ -\frac{\bar{F}_2(\delta)}{\bar{F}_3(\delta)} \end{bmatrix} = \begin{bmatrix} -2 - \delta^2 + 5\delta & 0 \\ 0 & 0 \\ 1 + 3\delta - 5\delta^2 + \delta^3 & 0 \\ 0 & 0 \\ 3 - 4\delta - \delta^2 + \delta^3 & 2\delta - 2 \\ 0 & 1 \end{bmatrix} \quad (27)$$

Step 3

Then we obtain the bicausal generalized change of coordinates $z = T(\delta)x$ where

$$T(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -1 & 2 - 4\delta + \delta^2 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -2 + 4\delta - \delta^2 & -\delta + \delta^2 & -1 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$$

with

$$[T(\delta)]_L^{-1} = \begin{bmatrix} 2 - 4\delta + \delta^2 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -19\delta^2 + 15\delta + 8\delta^3 - 4 - \delta^4 & 0 & -4\delta + 2 + \delta^2 & 0 & -1 & 0 \\ 15\delta^2 - 13\delta - 7\delta^3 + 4 + \delta^4 & 0 & 3\delta - 2 - \delta^2 & 1 & 1 & 0 \end{bmatrix}$$

Step 4

With the deduced change of coordinates, we have

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{\delta} & -\frac{1}{\delta} & -\frac{1}{\delta} & -\frac{1}{\delta} & 1 & -\frac{1}{\delta} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{\delta} & -\frac{1}{\delta} & -\frac{1}{\delta} & -\frac{1}{\delta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\bar{K}_1(\delta) \quad \bar{K}_2(\delta)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & \delta - 1 & 1 - \delta & 0 & 1 - \delta & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$[\bar{\Gamma}_1(\delta) \quad \bar{\Gamma}_2(\delta)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 4

For the simulation setting, we can choose

$$G_0 = \begin{bmatrix} 85 & 0 & 2000 & 0 & 12500 & 0 \\ 0 & 85 & 0 & 2000 & 0 & 12500 \end{bmatrix}^T$$

such that $(A_0 - G_0 C_0)$ has negative eigenvalues $(-10, -10, -25, -25, -50, -50)$.

Step 4

And finally we obtain the following gain matrices :

$$L_0 = \begin{bmatrix} -85 & 0 & 1 & 0 & 0 & 0 \\ 0 & -85 & 0 & 1 & 0 & 0 \\ -2000 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2000 & 0 & 0 & 0 & 1 \\ -12500 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12500 & 0 & 0 & 0 & 0 \end{bmatrix}, H(\delta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & \delta - 1 & 1 - \delta \\ 0 & 1 & -1 \end{bmatrix}$$
$$J(\delta) = \begin{bmatrix} 83 + 5\delta - \delta^2 & 0 & 0 \\ 0 & 0 & 0 \\ 2001 - 5\delta^2 + 3\delta + \delta^3 & \delta - 1 & 1 - \delta \\ 0 & 0 & 0 \\ 12503 - 4\delta - \delta^2 + \delta^3 & \delta - 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

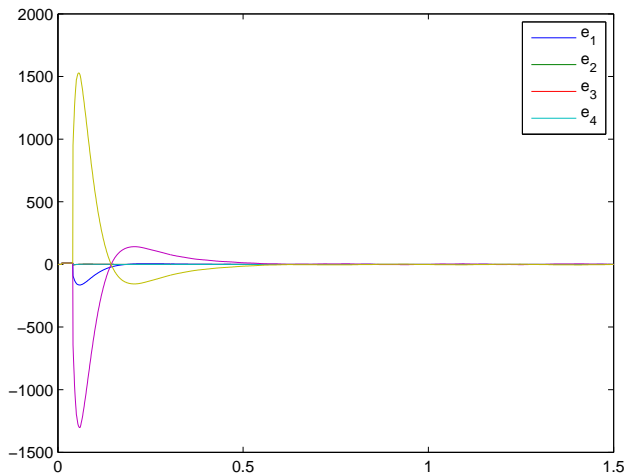


FIGURE : The observation error for $h=0.01s$.

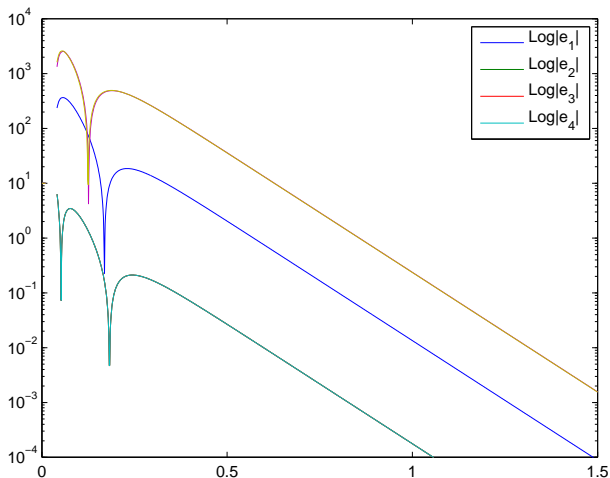
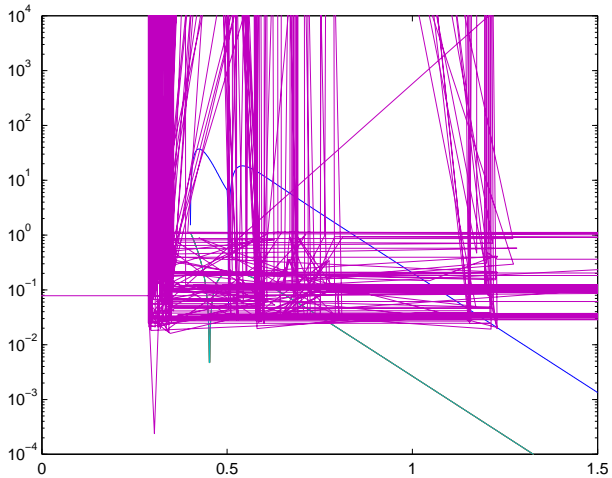


FIGURE : The observation error (in log scale) for $h = 0.01s$

Simulation



Thanks !

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