Observer synthesis under time-varying sampling for Lipschitz nonlinear systems

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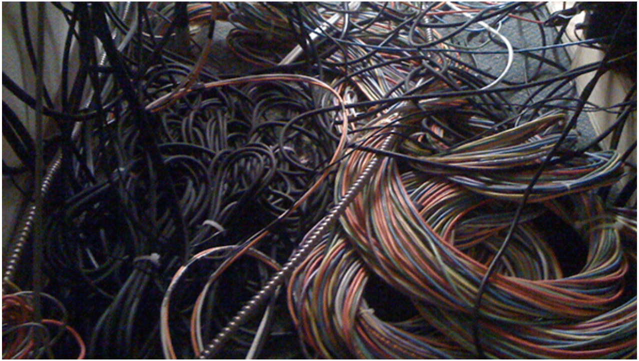
Motivations

- Technical reasons, economical reasons
- Relevant in the networked context



Problem formulation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + G\phi(Hx(t)), t \geq 0, \\ y_k &= Cx(t_k), k \in \mathbb{N}. \end{aligned} \quad (1)$$

- $y_k = Cx(t_k)$ the output available during $[t_k, t_{k+1})$.
- with $\phi : \mathbb{R}^m \to \mathbb{R}^m$ satisfying:

$$|\phi(a) - \phi(b)| \leq \gamma |a - b|, \quad \forall (a, b) \in \mathbb{R}^m \times \mathbb{R}^m, \quad (2)$$

  with $\gamma > 0$.
- $$S_{[\underline{\tau}, \overline{\tau}]} := \{(t_k)_{k\in\mathbb{N}} \; : \; t_0 = 0, \; t_{k+1} - t_k \in [\underline{\tau}, \overline{\tau}], k \in \mathbb{N}\}.$$

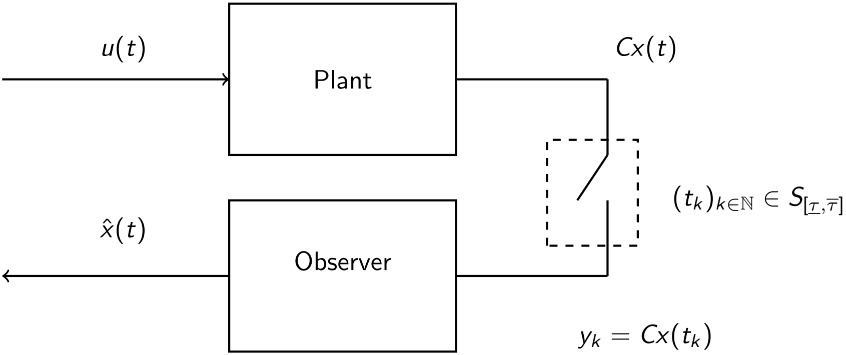


Figure : Structure of the system

Two main approaches

- $\dot{\hat{x}}(t) = F_1\big(\hat{x}(t), \hat{x}(t_k), y_k, u(t)\big)$
- $\dot{\hat{x}}(t) = F_2\big(\hat{x}(t), u(t)\big),\ \hat{x}(t_k^+) = R\big(\hat{x}(t_k^-), y_k\big)$

Structure of the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G\phi\left(H\hat{x}(t)\right), \quad t \in [t_k, t_{k+1}), \tag{3a}$$

$$\hat{x}(t_k) = \hat{x}(t_k^-) + K(t_k - t_{k-1})C\Big(x(t_k^-) - \hat{x}(t_k^-)\Big) \tag{3b}$$

- $\hat{x} \in \mathbb{R}^n$, observer state.
- $K : [\underline{\tau}, \overline{\tau}] \to \mathbb{R}^{n \times q}$ bounded and continuous

Observation error

$z = x - \hat{x}$, observation error.

$$\dot{z}(t) = Az(t) + G\left[\phi\Big(Hx(t)\Big) - \phi\Big(H\Big(x(t) - z(t)\Big)\Big)\right], \quad t \in [t_k, t_{k+1}), \tag{4a}$$

$$z(t_k) = (I - K(t_k - t_{k-1})C)z(t_k^-), \quad k \in \mathbb{N},\ k \geq 1, \tag{4b}$$

The solutions of both (3) and (4) are defined in an iterative way.

Different methodological approaches

- Embedding the non linearity and computing reachable sets (discrete system analysis): (Andrieu, Nadri Mazenc, Dinh)
  Leads to complex computations.
- Use Piece-wise continuous time varying Lyapunov function Hespanha, Raff, Chen
  $\Rightarrow$ Reformulation as an hybrid system.

Embedding the error dynamics

Lemma (Zemouche 2008)

Consider equations (1),(3),(4). Then, there exists a finite set of matrices R_i, $i \in \mathcal{P} := \{1, 2, \dots, p\}$, such that for any $(x, z) \in \mathbb{R}^n \times \mathbb{R}^n$

$$Az + G\left[\phi\Big(Hx\Big) - \phi\Big(H(x-z)\Big)\right] \in Cov\left\{R_i z\right\}_{i\in\mathcal{P}}.$$

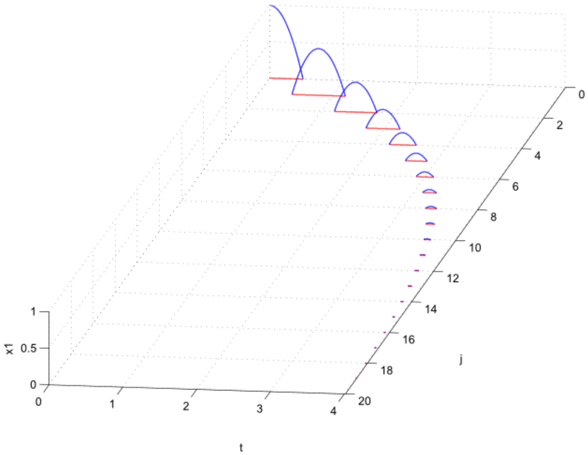
Hybrid reformulation of the observation problem

Consider a hybrid system

$$\mathcal{H} : \begin{cases} \xi \in C_{\mathcal{H}}, & \dot{\xi} \in F_{\mathcal{H}}(\xi), \\ \xi \in D_{\mathcal{H}}, & \xi^{+} \in G_{\mathcal{H}}(\xi). \end{cases} \qquad (5)$$

While ξ belongs to $C_{\mathcal{H}}$, the state *flows* according $F_{\mathcal{H}}$.
When ξ belongs to $D_{\mathcal{H}}$, the state *jumps* according $G_{\mathcal{H}}$.

Solution



Hybrid arc corresponding to a solution of the bouncing ball example: height

New hybrid system

$$\left.\begin{aligned} &\dot{z} \in Cov\{R_i z\}_{i\in\mathcal{P}} \\ &\dot{\tau} = 1 \\ &\dot{s} = -1 \end{aligned}\right\} (\tau, s) \in [0, \overline{\tau}]^2, \tag{6}$$

and

$$\left.\begin{aligned} &z^+ = (I - K(\tau)C)z \\ &\tau^+ = 0 \\ &s^+ \in [\underline{\tau}, \overline{\tau}] \end{aligned}\right\} (s = 0) \wedge (\tau \in [\underline{\tau}, \overline{\tau}]). \tag{7}$$

(6) dynamic in between sampling times, while (7) describes the impulsive dynamics.

- z the dynamics of the observation error
- $s \in [0, \overline{\tau}]$ (decreasing), gives the time before the next sampling occurs
- $\tau \in [0, \overline{\tau}]$ (increasing) accounts for the time elapsed since the last sampling.

- τ and s do not converge to 0 so convergence to a set $\mathcal{A}$ is considered.
- Only a subset of solutions to (6), (7) is considered.
- The concept of uniformly globally pre-asymptotically stable (UGpAS) is used.

Hybrid Lyapunov functions

Theorem (Sufficient conditions for UGpAS)

Let $\mathcal{H}$ be a hybrid system and let $\mathcal{A} \subset \mathbb{R}^n$ be closed. If V is a Lyapunov function candidate and $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$, continuous $\rho \in \mathcal{PD}$ with :

$$\forall \xi \in C_{\mathcal{H}} \cup D_{\mathcal{H}} \cup G_{\mathcal{H}}(D_{\mathcal{H}})$$

$$\alpha_1(|\xi|_{\mathcal{A}}) \leq V(\xi) \leq \alpha_2(|\xi|_{\mathcal{A}}),$$

$$\forall \xi \in C_{\mathcal{H}}, f \in F_{\mathcal{H}}(\xi), \langle \nabla V(\xi), f \rangle \leq -\rho(|\xi|_{\mathcal{A}}),$$

$$\forall \xi \in D_{\mathcal{H}}, g \in G_{\mathcal{H}}(\xi), V(g) - V(\xi) \leq -\rho(|\xi|_{\mathcal{A}}).$$

Then $\mathcal{A}$ is UGpAs for $\mathcal{H}$.

with $|\xi|_{\mathcal{A}}$ is the distance of ξ to $\mathcal{A}$

Formal definition

- flow set: $C_{\mathcal{H}} = \{\xi \in \mathbb{R}^{n+2} | z \in \mathbb{R}^n, (\tau, s) \in [0, \overline{\tau}]^2\}$;
- flow map: $F_{\mathcal{H}}(\xi) = \begin{pmatrix} Cov\{R_i z\}_{i \in \mathcal{P}} \\ 1 \\ -1 \end{pmatrix}$ for ξ in $C_{\mathcal{H}}$ and $F_{\mathcal{H}}(\xi) = \emptyset$ elsewhere;
- jump set: $D_{\mathcal{H}} = \{\xi \in \mathbb{R}^{n+2} |, z \in \mathbb{R}^n, (\tau \in [\underline{\tau}, \overline{\tau}]) \wedge (s = 0)\}$;
- jump map: $G_{\mathcal{H}}(\xi) = \begin{pmatrix} (I - K(\tau)C)z \\ 0 \\ [\underline{\tau}, \overline{\tau}] \end{pmatrix}$ for ξ in $D_{\mathcal{H}}$ and $G_{\mathcal{H}}(\xi) = \emptyset$ elsewhere.

Equivalence of solutions I

Consider z solution (4) with x satisfying (1) and $(t_k)_{k\in\mathbb{N}} \in S_{[\underline{\tau},\overline{\tau}]}$. Given $E = \cup_{k=0}^{\infty}([t_k, t_{k+1}], k)$, define the hybrid arc $\xi : E \to \mathbb{R}^{n+2}$ as follows

$$\xi(t,k) = \left(z'(t), t - t_k, t_{k+1} - t\right)', t \in [t_k, t_{k+1}), \tag{9}$$

and

$$\xi(t_{k+1},k) = \lim_{t \to t_{k+1}, t < t_{k+1}} \left(z'(t), t - t_k, t_{k+1} - t\right)'. \tag{10}$$

Using Lemma 1 and the description (6),(7), one can see that ξ is a complete solution to the hybrid system (5) with data $C_{\mathcal{H}}, F_{\mathcal{H}}, D_{\mathcal{H}}, G_{\mathcal{H}}$

$$\mathcal{A} = \{\xi \in C_{\mathcal{H}} \cup D_{\mathcal{H}} | z = 0, \tau \in [0, \overline{\tau}], s \in [0, \overline{\tau}]\}. \tag{11}$$

Problem

Problem: Consider system (6),(7). Design the gain function $K(\tau)$ such that the set $\mathcal{A}$ is UGpAS.

Theorem

Consider the hybrid system, (6),(7), a matrix function $P(\tau, s)$, $P : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$, *with* $\frac{\partial P}{\partial \tau}, \frac{\partial P}{\partial s}$ *being continuous on an open set containing* $[0, \overline{\tau}]^2$, *and the following set of LMIs*

$$P(\tau, s) = P(\tau, s)' > 0, \forall(\tau, s) \in [0, \overline{\tau}]^2, \tag{12a}$$

$$N_i(\tau, s) = \left(P(\tau, s)R_i\right) + \left(P(\tau, s)R_i\right)' + \frac{\partial P}{\partial \tau} - \frac{\partial P}{\partial s} < 0,$$
$$\forall(\tau, s) \in [0, \overline{\tau}]^2, \, \forall\, i \in \mathcal{P}, \tag{12b}$$

$$\left(\begin{array}{c|c} \begin{matrix} P(0, s)K(\tau)C \\ +P(\tau, 0) - P(0, s) \end{matrix} & C'K(\tau)'P(0, s) \\ \hline \star & P(0, s) \end{array}\right) > 0,$$
$$\forall(\tau, s) \in [\underline{\tau}, \overline{\tau}]^2. \tag{12c}$$

If the LMIs (12) are verified, then the set $\mathcal{A}$ *defined in (11) is UGpAS.*

Argument of proof

$$V(\xi) = z' P(\tau, s) z$$

- (12a) ensure positive definiteness of the Candidate Lyapunov function.
- (12b) ensure decrease during flow.
- (12c) ensure decrease during jump.

Recapturing other frameworks I

Link with a time delay point of view.
The Lyapunov function $V(\xi) = z^T P(\tau, s) z$ recaptures the structure of the Lyapunov-Krasovskii functional [Fridman 2010].

$$\bar{V}(t) = z'(t)\bar{P}z(t) + (t_{k+1} - t)\int_{t_k}^{t} \dot{z}'(\theta)U\dot{z}(\theta)d\theta \\ +(t_{k+1} - t)(z(t)', z(t_k)')Q(z(t)', z(t_k)')'.$$

Fridman 2010

E. Fridman. A refined input delay approach to sampled-data control. *Automatica*, Vol 46, No 2, pp. 421-427, 2010.

Recapturing other frameworks II

Pose $t - t_k = \tau$, $t_{k+1} - t = s$ as in our framework. Let

$$\mathcal{Q}(\tau) = \begin{pmatrix} I_d & 0 \\ 0 & e^{-A\tau} \end{pmatrix}' Q \begin{pmatrix} I_d & 0 \\ 0 & e^{-A\tau} \end{pmatrix}$$

and

$$M(\tau) = \int_0^\tau e^{\alpha(\rho-\tau)} (e^{A(\rho-\tau)})' A' U A e^{A(\rho-\tau)} d\rho.$$

The functional can be expressed as $\bar{V}(t) = z'(t) P(\tau(t), s(t)) z(t)$, where $P(\tau, s) = \bar{P} + s\mathcal{Q}(\tau) + M(\tau)$.

Theorem (Tractable conditions)

Consider the hybrid system (5), (6),(7). If there exist positive definite matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$ and matrices $L_0, L_1 \in \mathbb{R}^{n\times q}$ such that

$$M_p^1 = (P_1 R_p)' + (P_1 R_p) + \frac{P_2 - P_1}{\overline{\tau}} < 0, \; \forall p \in \mathcal{P}, \tag{14a}$$

$$M_p^2 = (P_2 R_p)' + (P_2 R_p) + \frac{P_2 - P_1}{\overline{\tau}} < 0, \; \forall p \in \mathcal{P}, \tag{14b}$$

$$\begin{pmatrix} (L_0 C) + (L_0 C)' - \frac{\underline{\tau}(P_1 - P_2)}{\overline{\tau}} & C' L_0' \\ \star & P_1 \end{pmatrix} > 0, \tag{14c}$$

$$\begin{pmatrix} (L_1 C)' + (L_1 C) - (P_1 - P_2) & C' L_1' \\ \star & P_1 \end{pmatrix} > 0, \tag{14d}$$

are verified, then the gain

$$K(\tau) = P_1^{-1}\left(L_0 + \frac{\tau - \underline{\tau}}{\overline{\tau} - \underline{\tau}}(L_1 - L_0)\right),$$

ensures that the set $\mathcal{A}$ defined in (11) is UGpAS.

Argument of proof

$$V(\xi) = z' P(\tau) z;\ P(\tau) = P_1 \left(1 - \frac{\tau}{\overline{T}}\right) + P_2 \frac{\tau}{\overline{T}}$$

- (14) $\Rightarrow$ (12) with $P(\tau, s) = P(\tau) = P_1 \left(1 - \frac{\tau}{\overline{T}}\right) + P_2 \frac{\tau}{\overline{T}}$
- Argument of convexity, Shur complement.

Minor modifications

- Periodic case: $\underline{\tau} = \overline{\tau}$, dropping (14d).
- Time varying sampling with static gain $L_1 = L_0$ in (14d).

Flexible joint: data of the system

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 21.6 & 0 & 0 \end{pmatrix}', \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

with $\phi = (0, 0, 0, 3.3 \sin x_3)'$. The input $u = sin(t)$ is applied to the system. Using the proposed conditions for $\underline{\tau} = 0.01, \overline{\tau} = 0.2$ one obtains the following gain:

$$K = \begin{pmatrix} 0.87 & 0.03 & 0.66 & 0.65 \\ 0.04 & 0.99 & 0.09 & -0.12 \end{pmatrix}'.$$

Flexible joint: static varying gain

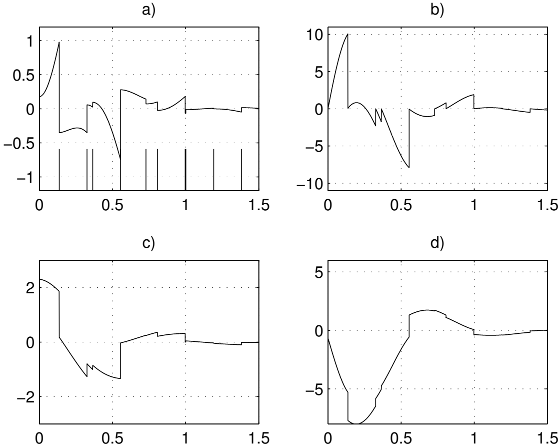
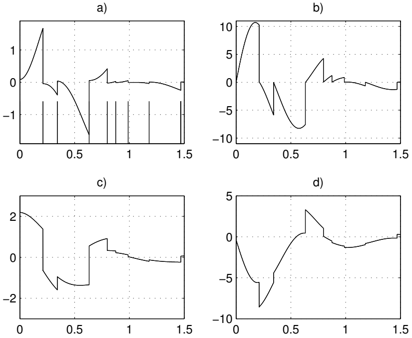


Figure : Observation error of the 4 states with static gain K. The sampling instants are indicated in plot a), bellow the evolution of z_1.

Flexible joint: time varying gain



$$K_0 = \begin{pmatrix} 0.92 & 0.00 & 0.77 & 0.4 \\ 0.00 & 1 & 0.04 & -0.06 \end{pmatrix}', K_1 = \begin{pmatrix} 0.98 & 0.01 & 0.91 & 0.94 \\ 0.01 & 1 & 0.05 & 0.01 \end{pmatrix}',$$

$$K(\tau) = K_0 + \frac{\tau - \underline{\tau}}{\overline{\tau} - \underline{\tau}}(K_1 - K_0)$$

DC drive: data of the system

Single-link direct-drive manipulator actuated by a permanent magnet DC brush motor:

$$\dot{x} = \begin{pmatrix} x_2 \\ -2sin(x_1) - 3x_2 + x_3 \\ -x_2 - x_3 \end{pmatrix}$$

$y(t_k) = (x_1(t_k), 0, 0)$. During flow

$$\dot{\hat{x}} = \begin{pmatrix} \hat{x}_2 \\ -sin(\hat{x}_1) - 3\hat{x}_2 + \hat{x}_3 \\ -\hat{x}_2 - \hat{x}_3 \end{pmatrix}.$$

At sampling time $\hat{x}^+ = \hat{x} + K(\tau)(x - \hat{x})$ $\underline{\tau} = 0.1, \overline{\tau} = 1$ With the following static gain $K = (1, 0.2228, -0.3226)^T$.

DC drive: static gain

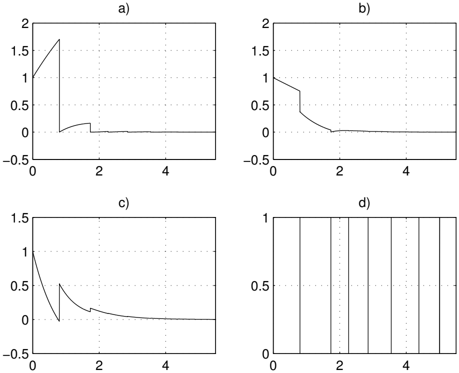


Figure : Observation error a) $x_1 - \hat{x}_1$, b) $x_2 - \hat{x}_2$, c)$x_3 - \hat{x}_3$, d) Sampling instants

Remark

Recalling the condition of (3) during Jump:

$$(I - KC)^T P(0)(I - KC) < P(\tau), \tau \in [\underline{\tau}, \overline{\tau}]$$

Similar conditions have been proposed to solve the problem of observer synthesis:

$$(I - KC)^T M_i^T P M_i (I - KC) - P, M_i \in \mathcal{D}$$

Where $M_i \in \mathcal{D}$ is an approximation of a reachable set

$$((I - KC)e^{A\tau})^T P(e^{A\tau}(I - KC)) - P, \tau \in [\underline{\tau}, \overline{\tau}]$$

Was proposed for linear systems (Here $e^{A\tau}$ is an exact computation of a reachable set...)
Those equations suggest that computing a $P(\tau)$ compatible with the decrease during flow allows to avoid or is somehow equivalent to computing reachable sets.

Extension: robustness

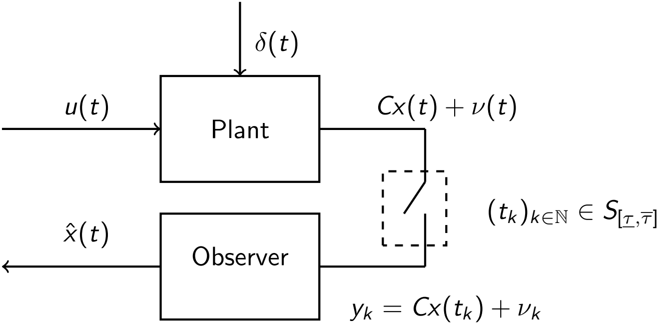


Figure : Structure of the observation scheme with noise and state perturbation

Extension: decentralized sensors

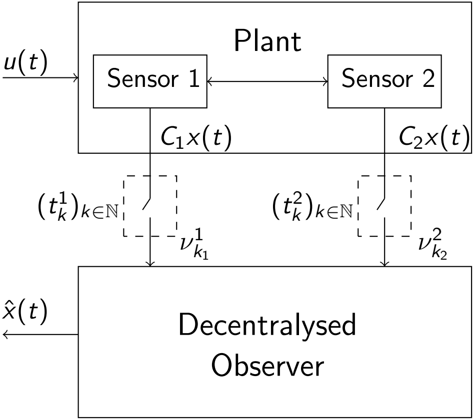


Figure : Structure of the observation scheme with decentralized sensors

Thank you for your attention.