Low-Power High-Gain Observers

Daniele Astolfi

Work (mainly) done in collaboration with L. Marconi and L. Praly.

Thanks also to A. Teel, A. Isidori and L. Wang for their hints and suggestions.

GT SYNC et Observation

JA MACS 16, November 16, 2016

We focus on systems of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= \varphi(x, d(t)) \\ y &= x_1 + \nu(t) \end{aligned}$$

but the results could be extended to more general triangular forms..

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, u) \\ &\vdots \\ \dot{x}_n &= f_3(x_1, x_2, \ldots, x_n, d(t), u) \\ y &= h(x_1, u) + \nu(t) \end{aligned}$$

We focus on systems of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= \varphi(x, d(t)) \\ y &= x_1 + \nu(t) \end{aligned}$$

but the results could be extended to more general triangular forms..

Assumptions:

- φ is a locally Lipschitz function;
- the state $x = (x_1, \ldots, x_n)$ evolves in a given compact set $X \subset \mathbb{R}^n$;
- $d(t)$ is some bounded signal which may represent uncertainties (parameter, model) or exogenous unknown inputs;
- $\nu(t)$ is a bounded measurement noise;

Goal: design a robust (semi-global) *tunable* observer

- ✓ asymptotic estimate in nominal conditions ($d(t) \equiv 0$ and $\nu(t) \equiv 0$);
- ✓ robust to uncertainties $d(t)$ and ISS w.r.t. measurement noise;
- ✓ the rate of convergence may be made arbitrarily fast;
- ? sensitivity to measurement noise: trade-off between fast convergence and noise attenuation;

The framework

We focus on systems of the form

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_n &= \varphi(x, d(t))\\ y &= x_1 + \nu(t)\end{aligned}$$

but the results could be extended to more general triangular forms..

Assumptions:

- φ is a locally Lipschitz function;
- the state $x = (x_1, \ldots, x_n)$ evolves in a given compact set $X \subset \mathbb{R}^n$;
- $d(t)$ is some bounded signal which may represent uncertainties (parameter, model) or exogenous unknown inputs;
- $\nu(t)$ is a bounded measurement noise;

Goal: design a robust (semi-global) *tunable* observer

✓ asymptotic estimate in nominal conditions ($d(t) \equiv 0$ and $\nu(t) \equiv 0$);

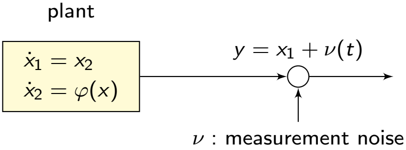
✓ robust to uncertainties $d(t)$ and ISS w.r.t. measurement noise;

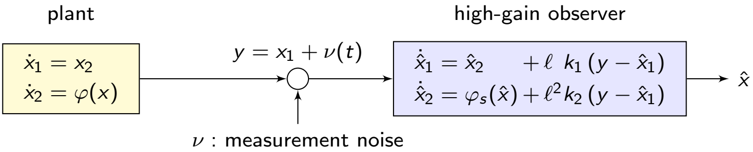
✓ the rate of convergence may be made arbitrarily fast;

? sensitivity to measurement noise: trade-off between fast convergence and noise attenuation;

Introduction to High-Gain Observers

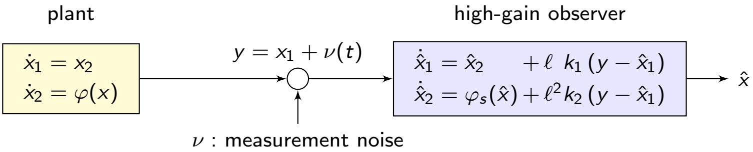
A simple example





- We refer to ℓ as the "high-gain parameter"
- $k_{,} k_2$ are parameters to be chosen
- $|\varphi(x) - \varphi_s(\hat{x})| \leq L|x - \hat{x}| + d$

A simple example



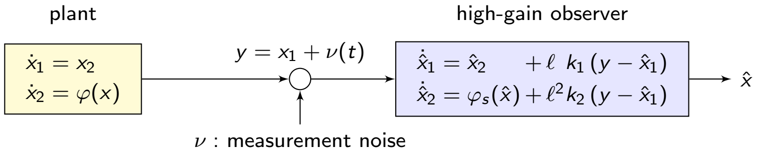
- We refer to ℓ as the "high-gain parameter"
- $k_, k_2$ are parameters to be chosen
- $|\varphi(x) - \varphi_s(\hat{x})| \le L|x - \hat{x}| + d$

Is the observer convergent?

Change of coordinates: $\quad e_1 := \hat{x}_1 - x_1\,, \quad e_2 := \dfrac{\hat{x}_2 - x_2}{\ell}$

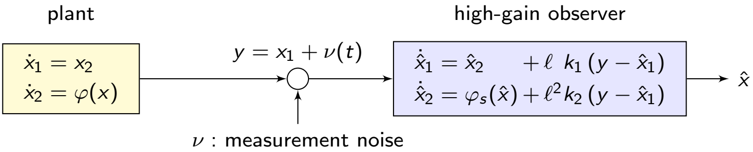
by which we obtain: $\quad \dot{e} \;=\; \ell\, A\, e + B\Delta(e) + \ell\, K\nu$

$$A = \begin{pmatrix} -k_1 & 1 \\ -k_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \quad |\Delta(e)| \le L|e| + \frac{1}{\ell} d\,.$$



If the "high-gain parameter" $\ell \geq 1$ is chosen large enough and $k_1 > 0$, $k_2 > 0$ then

$$|\hat{x}(t) - x(t)| \;\leq\; \alpha\,\ell\,\exp(-\sigma\,\ell\,t)|\hat{x}(0) - x(0)| + \frac{\mu}{\ell}\|d\|_\infty + \gamma\ell\|\nu\|_\infty$$

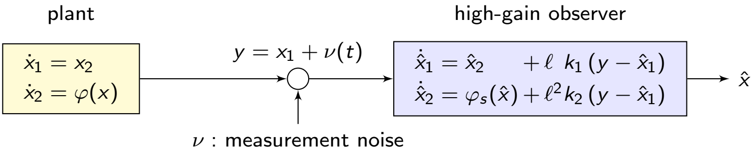


If the "high-gain parameter" $\ell \geq 1$ is chosen large enough and $k_1 > 0$, $k_2 > 0$ then

$$|\hat{x}(t) - x(t)| \;\leq\; \alpha\,\ell\,\exp(-\sigma\,\ell\,t)|\hat{x}(0) - x(0)| + \frac{\mu}{\ell}\|d\|_\infty + \gamma\ell\|\nu\|_\infty$$

- ✓ Asymptotic estimate when $d = 0$, $\nu = 0$
- ✓ Convergence arbitrarily fast: $\exp(-\sigma\,\ell\,t)$
- ✓ Robustness property: $\mu\|d\|_\infty/\ell$
- ✓ ISS: $\gamma\ell\|\nu\|_\infty$

A simple example



If the "high-gain parameter" $\ell \geq 1$ is chosen large enough and $k_1 > 0$, $k_2 > 0$ then

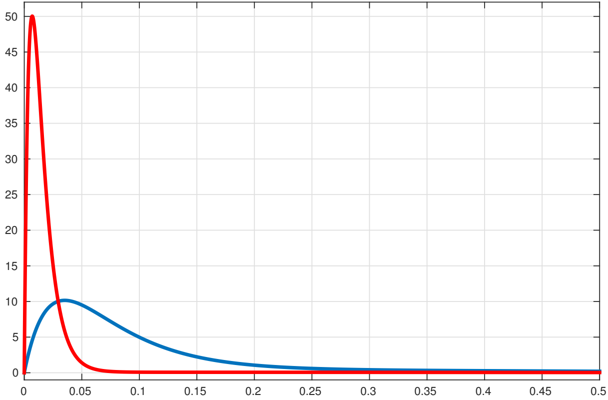
$$|\hat{x}(t) - x(t)| \;\leq\; \alpha \, \ell \exp(-\sigma \, \ell \, t)|\hat{x}(0) - x(0)| + \frac{\mu}{\ell}\|d\|_\infty + \gamma\ell\|\nu\|_\infty$$

- ✓ Asymptotic estimate when $d = 0$, $\nu = 0$
- ✓ Convergence arbitrarily fast: $\exp(-\sigma \, \ell \, t)$
- ✓ Robustness property: $\mu\|d\|_\infty/\ell$ ✓ ISS: $\gamma\ell\|\nu\|_\infty$

- ✗ Peaking phenomenon: $\alpha \, \ell|\hat{x}(0) - x(0)|$

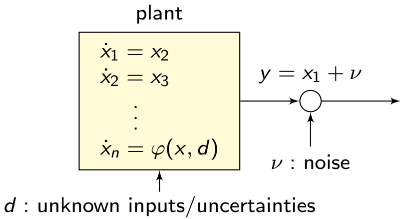
A simple example: the peaking phenomenon

$$|\hat{x}_2(t) - x_2(t)| \quad \leq \quad \alpha\,\ell\,\exp(-\sigma\,\ell\,t)\;|\hat{x}(0) - x(0)| + \frac{\mu}{\ell}\,\|d\|_\infty$$



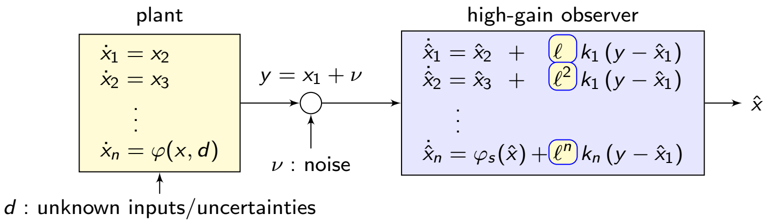
- $\ell = 20$
- $\ell = 100$

High-gain observers in a general framework



- $x = (x_1, \ldots, x_n)^\top$ evolves in a compact set $X \subset \mathbb{R}^n$
- φ Locally Lipschitz

High-gain observers in a general framework

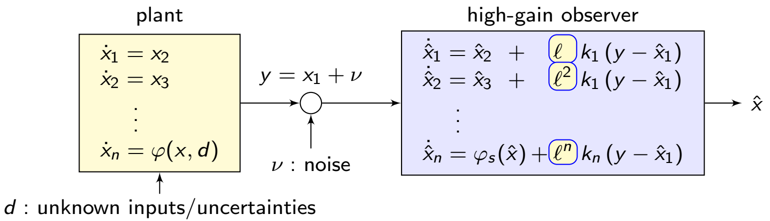


- $x = (x_1, \ldots, x_n)^\top$ evolves in a compact set $X \subset \mathbb{R}^n$
- φ Locally Lipschitz

- $\lambda^n + k_1 \lambda^{n-1} + \ldots + k_{n-1}\lambda + k_n$ Hurwitz
- $\ell \geq 1$ high-gain parameter
- $|\varphi(x, d) - \varphi_s(\hat{x})| \;\leq\; L_\varphi |x - \hat{x}| + R\,|d|$ for all $(x, \hat{x})$ in $X \times \mathbb{R}^n$, $L_\varphi > 0$, $R > 0$

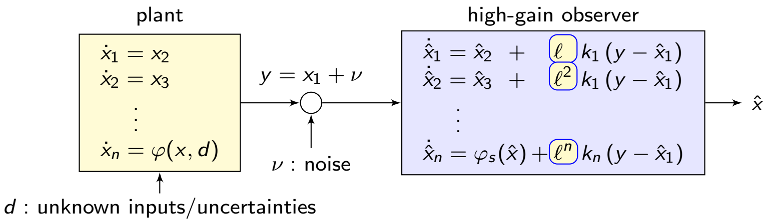
Remark: with $\varphi_s = 0$ we get a linear “rough” differentiator (also called dirty - derivative observer)

Highlights on high-gain observer: summary



$$|\hat{x}(t) - x(t)| \;\leq\; \alpha\,\ell^{n-1}\exp(-\sigma\,\ell\,t)|\hat{x}(0) - x(0)| + \frac{\mu}{\ell}\|d\|_\infty + \gamma\ell^{n-1}\|\nu\|_\infty$$

- ✓ Asymptotic estimate for ℓ large enough if $d = 0$, $\nu = 0$
- ✓ Tunability property: $\exp(-\sigma\,\ell\,t)$
- ✓ Robustness property: $\mu\|d\|_\infty/\ell$
- ✓ ISS property: $\gamma\ell^{n-1}\|\nu\|_\infty$



$$|\hat{x}(t) - x(t)| \leq \alpha \, \ell^{n-1} \exp(-\sigma \, \ell \, t)|\hat{x}(0) - x(0)| + \frac{\mu}{\ell}\|d\|_\infty + \gamma \ell^{n-1}\|\nu\|_\infty$$

- ✓ Asymptotic estimate for ℓ large enough if $d = 0$, $\nu = 0$
- ✓ Tunability property: $\exp(-\sigma \, \ell \, t)$
- ✓ Robustness property: $\mu\|d\|_\infty/\ell$
- ✓ ISS property: $\gamma\ell^{n-1}\|\nu\|_\infty$

- ✗ Peaking phenomenon ($\alpha \, \ell^{n-1}|\hat{x}(0) - x(0)|$) and numerical implementation (of constants and internal variables) when n (system dimension) is large
- ✗ Bad sensitivity to high-frequency measurement noise: $\bar{\gamma} \, \ell^n \, \|\nu\|_\infty$

A simple motivation behind the choice of the gains

Observed system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= L\, x_n \\ y &= x_1 \end{aligned}$$

Luenberger Observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + a_1(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + a_2(y - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + a_{n-1}(y - \hat{x}_1) \\ \dot{\hat{x}}_n &= L\, \hat{x}_n + a_n(y - \hat{x}_1) \end{aligned}$$

$$e = \hat{x} - x\,, \qquad \dot{e} = A\, e$$

$$A = \begin{pmatrix} -a_1 & 1 & 0 & \dots & \dots & 0 \\ -a_2 & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -a_{n-2} & 0 & \dots & 0 & 1 & 0 \\ -a_{n-1} & 0 & \dots & \dots & 0 & 1 \\ -a_n & 0 & \dots & \dots & \dots & L \end{pmatrix}$$

Convergence of the observer is guaranteed only if the matrix A is Hurwitz.

Observed system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_{n-1} &= x_n\\ \dot{x}_n &= L\,x_n\\ y &= x_1\end{aligned}$$

Luenberger Observer

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + a_1(y-\hat{x}_1)\\ \dot{\hat{x}}_2 &= \hat{x}_3 + a_2(y-\hat{x}_1)\\ &\vdots\\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + a_{n-1}(y-\hat{x}_1)\\ \dot{\hat{x}}_n &= L\,\hat{x}_n + a_n(y-\hat{x}_1)\end{aligned}$$

$$e = \hat{x} - x\,, \qquad \dot{e} = A\,e$$

$$A = \begin{pmatrix} -a_1 & 1 & 0 & \dots & \dots & 0\\ -a_2 & 0 & 1 & \ddots & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ -a_{n-2} & 0 & \dots & 0 & 1 & 0\\ -a_{n-1} & 0 & \dots & \dots & 0 & 1\\ -a_n & 0 & \dots & \dots & \dots & L \end{pmatrix}$$

Convergence of the observer is guaranteed only if the matrix A is Hurwitz.

A simple motivation behind the choice of the gains

Observed system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= L\, x_n \\ y &= x_1 \end{aligned}$$

Luenberger Observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + a_1(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + a_2(y - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + a_{n-1}(y - \hat{x}_1) \\ \dot{\hat{x}}_n &= L\, \hat{x}_n + a_n(y - \hat{x}_1) \end{aligned}$$

$$e = \hat{x} - x\,, \qquad \dot{e} = A\, e$$

The polynomial characteristic of A is given by

$$\lambda^n + (a_1 - L)\lambda^{n-1} + (a_2 - a_1 L)\lambda^{n-2} + \ldots + (a_{n-1} - a_{n-2}L)\lambda + (a_n - a_{n-1}L)$$

So a necessary condition for stability of the e-dynamics is

$$a_1 > L\,, \quad a_2 > a_1\, L\,, \quad \ldots\,, \quad a_{n-1} > a_{n-2}\, L\,, \quad a_n > a_{n-1}\, L$$

which implies $\boxed{a_i > L^i}$

✗ the gains must grow with powers of L!

- the choice $a_i = k_i\, \ell^i$ with $\ell > L$ is conservative, but the tuning is easy!

Observed system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= L\,x_n \\ y &= x_1 \end{aligned}$$

Luenberger Observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + a_1(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + a_2(y - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + a_{n-1}(y - \hat{x}_1) \\ \dot{\hat{x}}_n &= L\,\hat{x}_n + a_n(y - \hat{x}_1) \end{aligned}$$

$$e = \hat{x} - x\,, \qquad \dot{e} = A\,e$$

The polynomial characteristic of A is given by

$$\lambda^n + (a_1 - L)\lambda^{n-1} + (a_2 - a_1 L)\lambda^{n-2} + \ldots + (a_{n-1} - a_{n-2}L)\lambda + (a_n - a_{n-1}L)$$

So a necessary condition for stability of the e-dynamics is

$$a_1 > L\,, \qquad a_2 > a_1\,L\,, \qquad \ldots\,, \qquad a_{n-1} > a_{n-2}\,L\,, \qquad a_n > a_{n-1}\,L$$

which implies $\boxed{a_i > L^i}$

- ✗ the gains must grow with powers of L!
- the choice $a_i = k_i\,\ell^i$ with $\ell > L$ is conservative, but the tuning is easy!

Benefits:

- easy to implement
- tunability property with ℓ
- robust to uncertainties

Drawbacks:

- ✗ the state of the system $\hat{x}$ may exit out of the compact set X
- ✗ numerical issues: implementation of constants of order $O(\ell^n)$ and variables of order $O(\ell^{n-1})$ during the transient
- ✗ sensitivity to measurement noise

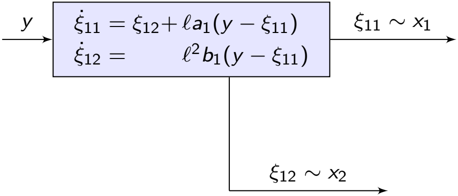
Question: Can we design an observer which maintains the same "good" features while improving the other drawbacks ?

Introduction to low-power high-gain observers

Cascade of 2 dimensional high-gain observers

plant

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_n &= \varphi(x,d)\\ y &= x_1\end{aligned}$$

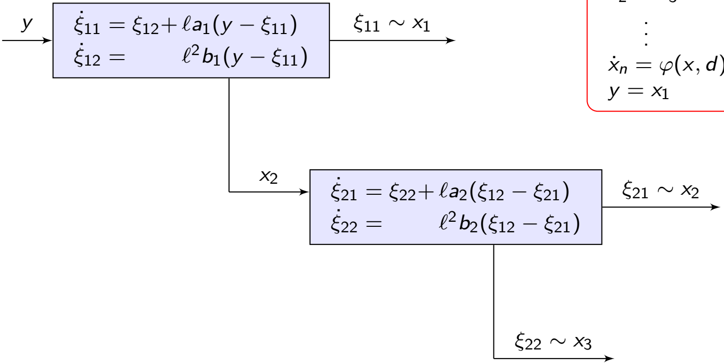


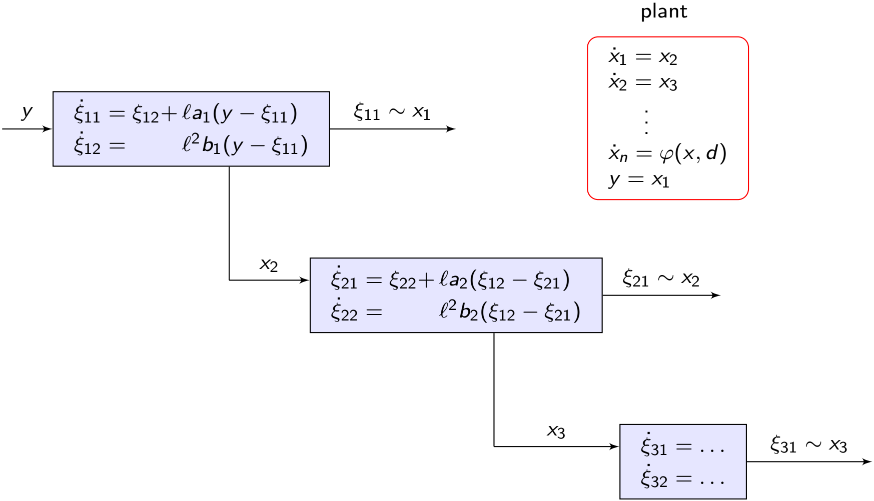
plant

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_n &= \varphi(x, d) \\
y &= x_1
\end{aligned}
$$

plant

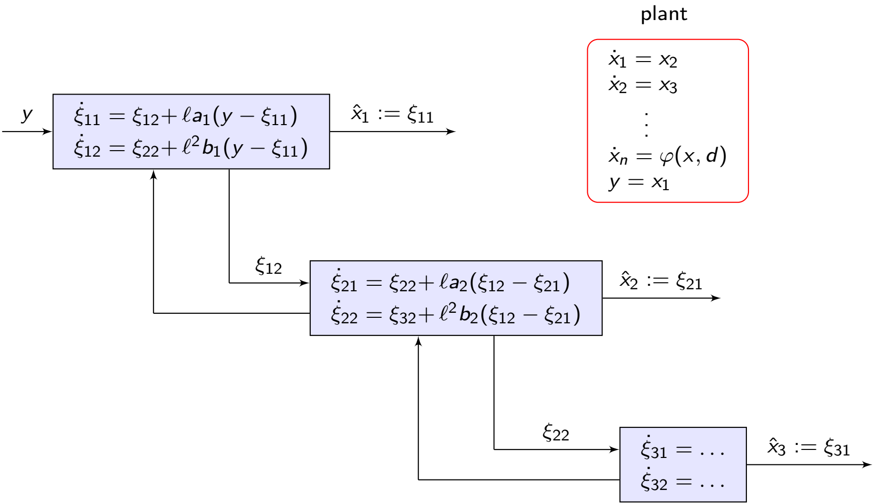
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_n &= \varphi(x,d)\\ y &= x_1\end{aligned}$$





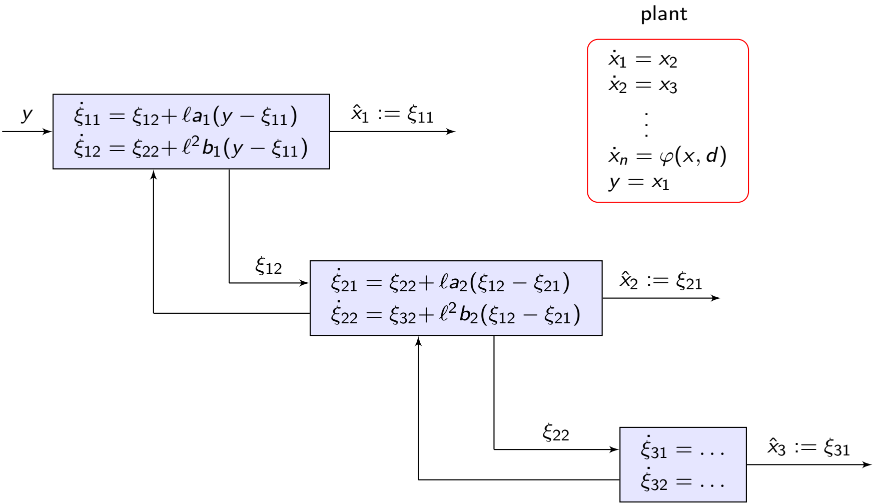
✗ note that with this design we cannot achieve asymptotic estimation!

the *consistency* terms are missing!



- we add the interconnection terms
- for a system of dimension n we need $n-1$ blocks of dimension 2!

The *low-power* high-gain construction



- we add the interconnection terms
- for a system of dimension n we need $n - 1$ blocks of dimension 2!

Low-power high-gain observer for a system of dimension n

$$
\begin{aligned}
\dot{\xi}_{11} &= \xi_{12} + \ell\, a_1\,(y - \xi_{11}) \\
\dot{\xi}_{12} &= \xi_{22} + \ell^2\, b_1\,(y - \xi_{11}) \\
&\vdots \\
\dot{\xi}_{i1} &= \xi_{i2} + \ell\, a_i\,(\xi_{(i-1)2} - \xi_{i1}) \\
\dot{\xi}_{i2} &= \xi_{(i+1)2} + \ell^2\, b_i\,(\xi_{(i-1)2} - \xi_{i1}) \\
&\vdots \\
\dot{\xi}_{(n-1)1} &= \xi_{(n-1)2} + \ell\, a_{n-1}\,(\xi_{(n-2)2} - \xi_{(n-1)1}) \\
\dot{\xi}_{(n-1)2} &= \varphi_s(\hat{x}) + \ell^2\, b_{n-1}\,(\xi_{(n-2)2} - \xi_{(n-1)1})
\end{aligned}
\qquad
\hat{x} := \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{i1} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix}
\quad
\xi = \begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \vdots \\ \xi_{i1} \\ \xi_{i2} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix}
$$

If (a_i, b_i) are properly chosen and ℓ is large enough then :

$$
|\hat{x}(t) - x(t)| \;\le\; \alpha\,\ell^{n-1}\exp(-\sigma\,\ell\,t)|\hat{x}(0) - x(0)| + \frac{\mu}{\ell}\|d\|_\infty + \gamma\ell^{n-1}\|\nu\|_\infty
$$

- ✓ we obtained the same asymptotic bounds of a standard high-gain observer by implementing only terms in ℓ and ℓ^2
- ✓ the relative degree improves the performances in presence of high-frequency measurement noise

$$
\begin{aligned}
\dot{\xi}_{11} &= \xi_{12} + \ell\, a_1\,(y - \xi_{11}) \\
\dot{\xi}_{12} &= \xi_{22} + \ell^2\, b_1\,(y - \xi_{11}) \\
&\vdots \\
\dot{\xi}_{i1} &= \xi_{i2} + \ell\, a_i\,(\xi_{(i-1)2} - \xi_{i1}) \\
\dot{\xi}_{i2} &= \xi_{(i+1)2} + \ell^2\, b_i\,(\xi_{(i-1)2} - \xi_{i1}) \\
&\vdots \\
\dot{\xi}_{(n-1)1} &= \xi_{(n-1)2} + \ell\, a_{n-1}\,(\xi_{(n-2)2} - \xi_{(n-1)1}) \\
\dot{\xi}_{(n-1)2} &= \varphi_s(\hat{x}) + \ell^2\, b_{n-1}\,(\xi_{(n-2)2} - \xi_{(n-1)1})
\end{aligned}
$$

$$
\hat{x} := \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{i1} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix} \qquad \xi = \begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \vdots \\ \xi_{i1} \\ \xi_{i2} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix}
$$

If (a_i, b_i) are properly chosen and ℓ is large enough then :

$$
|\hat{x}(t) - x(t)| \;\leq\; \alpha\,\ell^{n-1}\exp(-\sigma\,\ell\,t)|\hat{x}(0) - x(0)| + \frac{\mu}{\ell}\|d\|_\infty + \gamma\ell^{n-1}\|\nu\|_\infty
$$

- ✓ we obtained the same asymptotic bounds of a standard high-gain observer by implementing only terms in ℓ and ℓ^2
- ✓ the relative degree improves the performances in presence of high-frequency measurement noise

Low-power high-gain observer for a system of dimension n

$$
\begin{aligned}
\dot{\xi}_{11} &= \xi_{12} + \boxed{\ell}\, a_1\,(y - \xi_{11}) \\
\dot{\xi}_{12} &= \xi_{22} + \boxed{\ell^2}\, b_1\,(y - \xi_{11}) \\
&\vdots \\
\dot{\xi}_{i1} &= \xi_{i2} + \boxed{\ell}\, a_i\,(\xi_{(i-1)2} - \xi_{i1}) \\
\dot{\xi}_{i2} &= \xi_{(i+1)2} + \boxed{\ell^2}\, b_i\,(\xi_{(i-1)2} - \xi_{i1}) \\
&\vdots \\
\dot{\xi}_{(n-1)1} &= \xi_{(n-1)2} + \boxed{\ell}\, a_{n-1}\,(\xi_{(n-2)2} - \xi_{(n-1)1}) \\
\dot{\xi}_{(n-1)2} &= \varphi_s(\hat{x}) + \boxed{\ell^2}\, b_{n-1}\,(\xi_{(n-2)2} - \xi_{(n-1)1})
\end{aligned}
$$

$$
\hat{x} := \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{i1} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix} \qquad \xi = \begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \vdots \\ \xi_{i1} \\ \xi_{i2} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix}
$$

If (a_i, b_i) are properly chosen and ℓ is large enough then :

$$
|\hat{x}(t) - x(t)| \;\leq\; \alpha\, \boxed{\ell^{n-1}} \exp(-\sigma\, \ell\, t)|\hat{x}(0) - x(0)| + \frac{\mu}{\ell}\|d\|_\infty + \gamma \ell^{n-1}\|\nu\|_\infty
$$

Open questions

- ✗ design of parameters (a_i, b_i)
- ✗ noise analysis
- ✗ peaking is still present

Standard high-gain observer: design of the parameters

For a **standard high-gain observer** the parameters k_i, $i = 1, \dots, n$ must be chosen such that the matrix F is Hurwitz

$$F = \begin{pmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ -k_{n-1} & & & & 1 \\ -k_n & 0 & \cdots & \cdots & 0 \end{pmatrix}_{n \times n}$$

- the eigenvalues of F can be arbitrarily assigned
- the minimum value of ℓ which guarantees convergence is $\ell^\star = 2L_\varphi \|P\|$ with

$$PF + F^\top P = -I$$

with L_φ Lipschitz constant of $\varphi(\cdot)$

For a **standard high-gain observer** the parameters k_i, $i = 1, \ldots, n$ must be chosen such that the matrix F is Hurwitz

$$F = \begin{pmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ -k_{n-1} & & & & 1 \\ -k_n & 0 & \cdots & \cdots & 0 \end{pmatrix}_{n \times n}$$

- the eigenvalues of F can be arbitrarily assigned
- the minimum value of ℓ which guarantees convergence is $\ell^\star = 2L_\varphi \|P\|$ with

$$PF + F^\top P = -I$$

with L_φ Lipschitz constant of $\varphi(\cdot)$

Standard high-gain observer: design of the parameters

For a **standard high-gain observer** the parameters k_i, $i = 1, \ldots, n$ must be chosen such that the matrix F is Hurwitz

$$F = \begin{pmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ -k_{n-1} & & & & 1 \\ -k_n & 0 & \cdots & \cdots & 0 \end{pmatrix}_{n\times n}$$

- the eigenvalues of F can be arbitrarily assigned
- the minimum value of ℓ which guarantees convergence is $\boxed{\ell^\star = 2L_\varphi \|P\|}$ with

$$PF + F^\top P = -I$$

with L_φ Lipschitz constant of $\varphi(\cdot)$

Likewise, for the **low-power high-gain observer** the parameters (a_i, b_i) must be chosen such that M is Hurwitz

$$M = \begin{pmatrix} E_1 & N & 0 & & \dots & \dots & 0 \\ Q_2 & E_2 & N & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & Q_i & E_i & N & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & Q_{n-2} & E_{n-2} & N \\ 0 & \dots & \dots & \dots & 0 & Q_{n-1} & E_{n-1} \end{pmatrix}_{(2n-2)\times(2n-2)}$$

$$E_i = \begin{pmatrix} -a_i & 1 \\ -b_i & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & a_i \\ 0 & b_i \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

✓ the eigenvalues of M can be arbitrarily assigned (with a recursive algorithm)

✓ the minimum value of ℓ which guarantees convergence is $\boxed{\ell^\star = 2L_\varphi \|P\|}$ with

$$PM + M^\top P = -I$$

Likewise, for the **low-power high-gain observer** the parameters (a_i, b_i) must be chosen such that M is Hurwitz

$$M = \begin{pmatrix} E_1 & N & 0 & & \dots & \dots & 0 \\ Q_2 & E_2 & N & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & Q_i & E_i & N & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & Q_{n-2} & E_{n-2} & N \\ 0 & \dots & \dots & \dots & 0 & Q_{n-1} & E_{n-1} \end{pmatrix}_{(2n-2)\times(2n-2)}$$

$$E_i = \begin{pmatrix} -a_i & 1 \\ -b_i & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & a_i \\ 0 & b_i \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

✓ the eigenvalues of M can be arbitrarily assigned (with a recursive algorithm)

✓ the minimum value of ℓ which guarantees convergence is $\boxed{\ell^\star = 2L_\varphi \|P\|}$ with

$$PM + M^\top P = -I$$

Low-power high-gain observer: design of the parameters

Likewise, for the **low-power high-gain observer** the parameters (a_i, b_i) must be chosen such that M is Hurwitz

$$M = \begin{pmatrix} E_1 & N & 0 & & \dots & \dots & 0 \\ Q_2 & E_2 & N & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & Q_i & E_i & N & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & Q_{n-2} & E_{n-2} & N \\ 0 & \dots & \dots & \dots & 0 & Q_{n-1} & E_{n-1} \end{pmatrix}_{(2n-2)\times(2n-2)}$$

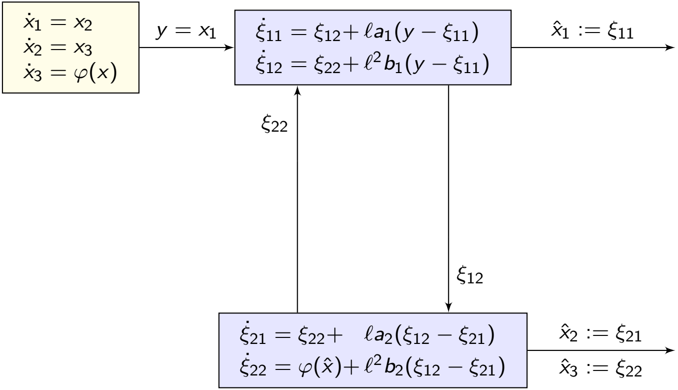
$$E_i = \begin{pmatrix} -a_i & 1 \\ -b_i & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & a_i \\ 0 & b_i \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- ✓ the eigenvalues of M can be arbitrarily assigned (with a recursive algorithm)
- ✓ the minimum value of ℓ which guarantees convergence is $\boxed{\ell^\star = 2L_\varphi \|P\|}$ with

$$PM + M^\top P = -I$$

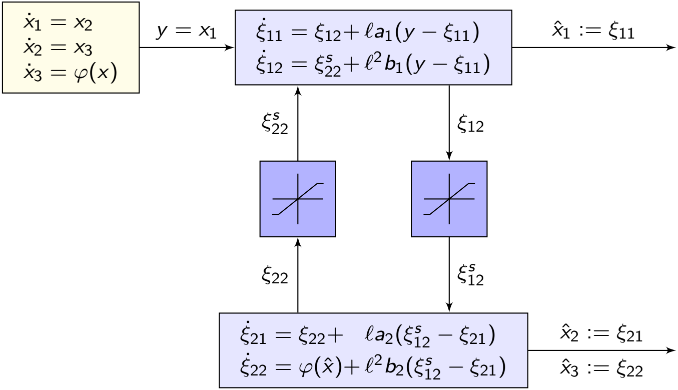
Peaking-free design of low-power high-gain observers

Low-power peaking-free design: an example



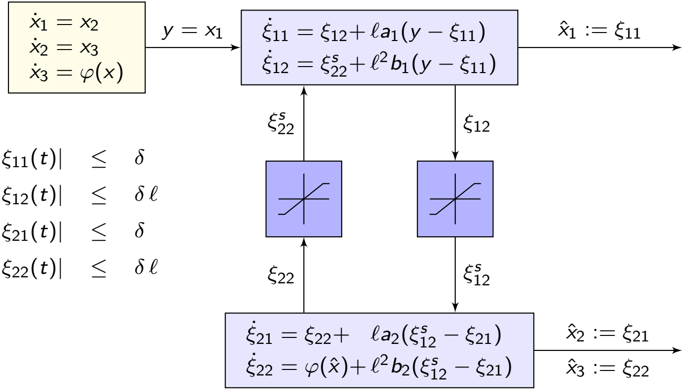
Peaking Phenomenon : $\quad |\hat{x}(t) - x(t)| \le \alpha \, \boxed{\ell^2} \, \exp(-\sigma \ell \, t) \, |\hat{x}(0) - x(0)|$

Low-power peaking-free design: an example



Main idea : add *saturations* between the blocks

The saturation levels are chosen coherently with the compact set X where the state of the system evolves



$$
\begin{aligned}
|\xi_{11}(t)| &\leq \delta \\
|\xi_{12}(t)| &\leq \delta\,\ell \\
|\xi_{21}(t)| &\leq \delta \\
|\xi_{22}(t)| &\leq \delta\,\ell
\end{aligned}
$$

We do not have anymore peaking in ℓ^2!

$$\begin{aligned}
\dot{\xi}_{11} &= \xi_{12} + \ell\ a_1\,(y - \xi_{11})\\
\dot{\xi}_{12} &= \xi_{22} + \ell^2\, b_1\,(y - \xi_{11})\\
&\vdots\\
\dot{\xi}_{i1} &= \xi_{i2} \quad + \ell\ a_i\,(\xi_{(i-1)2} - \xi_{i1})\\
\dot{\xi}_{i2} &= \xi_{(i+1)2} + \ell^2\, b_i\,(\xi_{(i-1)2} - \xi_{i1})\\
&\vdots\\
\dot{\xi}_{(n-1)1} &= \xi_{(n-1)2} + \ell\ a_{n-1}\,(\xi_{(n-2)2} - \xi_{(n-1)1})\\
\dot{\xi}_{(n-1)2} &= \varphi_s(\hat{x}) \ + , \ell^2\, b_{n-1}\,(\xi_{(n-2)2} - \xi_{(n-1)1})
\end{aligned}
\qquad
\hat{x} := \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{i1} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix}
\quad
\xi = \begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \vdots \\ \xi_{i1} \\ \xi_{i2} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix}$$

If (a_i, b_i) are properly chosen and ℓ is large enough then :

$$\boxed{|\hat{x}(t) - x(t)| \ \leq\ \alpha\,\ell^{n-1} \exp(-\sigma\,\ell\,t)|\hat{x}(0) - x(0)|}$$

$$
\begin{aligned}
\dot\xi_{11} &= \xi_{12} + \ell\, a_1 (y - \xi_{11}) \\
\dot\xi_{12} &= \boxed{\xi_{22}^s} + \ell^2 b_1 (y - \xi_{11}) \\
&\vdots \\
\dot\xi_{i1} &= \xi_{i2} + \ell\, a_i (\boxed{\xi_{(i-1)2}^s} - \xi_{i1}) \\
\dot\xi_{i2} &= \boxed{\xi_{(i+1)2}^s} + \ell^2 b_i (\boxed{\xi_{(i-1)2}^s} - \xi_{i1}) \\
&\vdots \\
\dot\xi_{(n-1)1} &= \xi_{(n-1)2} + \ell\, a_{n-1} (\boxed{\xi_{(n-2)2}^s} - \xi_{(n-1)1}) \\
\dot\xi_{(n-1)2} &= \varphi_s(\hat x) + \ell^2 b_{n-1} (\boxed{\xi_{(n-2)2}^s} - \xi_{(n-1)1})
\end{aligned}
\qquad
\hat x := \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{i1} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix}
\qquad
\xi = \begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \vdots \\ \xi_{i1} \\ \xi_{i2} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix}
$$

If (a_i, b_i) are properly chosen and ℓ is large enough then :

$$
|\hat x(t) - x(t)| \;\leq\; \min\left\{\alpha\, \ell^{n-1} \exp(-\sigma\, \ell\, t)|\hat x(0) - x(0)|\;,\; \boxed{\delta\ell}\right\}
$$

✓ the peaking does not grow with ℓ^{n-1} but only with ℓ

$$\begin{aligned}
\dot{\xi}_{11} &= \xi_{12} + \ell\ a_1 (y - \xi_{11}) \\
\dot{\xi}_{12} &= \xi_{22}^s + \ell^2 b_1 (y - \xi_{11}) \\
&\vdots \\
\dot{\xi}_{i1} &= \xi_{i2} + \ell\ a_i (\xi_{(i-1)2}^s - \xi_{i1}) \\
\dot{\xi}_{i2} &= \xi_{(i+1)2}^s + \ell^2 b_i (\xi_{(i-1)2}^s - \xi_{i1}) \\
&\vdots \\
\dot{\xi}_{(n-1)1} &= \xi_{(n-1)2} + \ell\ a_{n-1} (\xi_{(n-2)2}^s - \xi_{(n-1)1}) \\
\dot{\xi}_{(n-1)2} &= \varphi_s(\hat{x}) + \ell^2 b_{n-1} (\xi_{(n-2)2}^s - \xi_{(n-1)1})
\end{aligned}$$

$$\hat{x} := \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{i1} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix} \qquad \xi = \begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \vdots \\ \xi_{i1} \\ \xi_{i2} \\ \vdots \\ \xi_{(n-1)1} \\ \xi_{(n-1)2} \end{pmatrix}$$

If (a_i, b_i) are properly chosen and ℓ is large enough then :

$$|\hat{x}(t) - x(t)| \ \leq \ \min\left\{\alpha\,\ell^{n-1}\exp(-\sigma\,\ell\,t)|\hat{x}(0) - x(0)|\ ,\ \delta\ell\right\}$$

- ✓ the peaking does not grow with ℓ^{n-1} but only with ℓ
- ✓ the first $n-1$ estimates $\hat{x}_i := \xi_{i1}$ are peaking-free

$$
\begin{aligned}
\dot{\xi}_{11} &= \xi_{12} + \ell\ a_1 (y - \xi_{11}) \\
\dot{\xi}_{12} &= \xi_{22}^s + \ell^2 b_1 (y - \xi_{11}) \\
&\vdots \\
\dot{\xi}_{i1} &= \xi_{i2} + \ell\ a_i (\xi_{(i-1)2}^s - \xi_{i1}) \\
\dot{\xi}_{i2} &= \xi_{(i+1)2}^s + \ell^2 b_i (\xi_{(i-1)2}^s - \xi_{i1}) \\
&\vdots \\
\dot{\xi}_{(n-1)1} &= \xi_{(n-1)2} + \ell\ a_{n-1} (\xi_{(n-2)2}^s - \xi_{(n-1)1}) \\
\dot{\xi}_{(n-1)2} &= \varphi_s(\hat{x}) + \ell^2 b_{n-1} (\xi_{(n-2)2}^s - \xi_{(n-1)1})
\end{aligned}
\qquad
\hat{x} := \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{i1} \\ \vdots \\ \xi_{(n-1)1} \\ \boxed{\xi_{(n-1)2}} \end{pmatrix}
\qquad
\xi = \begin{pmatrix} \xi_{11} \\ \boxed{\xi_{21}} \\ \vdots \\ \xi_{i1} \\ \boxed{\xi_{i2}} \\ \vdots \\ \xi_{(n-1)1} \\ \boxed{\xi_{(n-1)2}} \end{pmatrix}
$$

If (a_i, b_i) are properly chosen and ℓ is large enough then :

$$
|\hat{x}(t) - x(t)| \leq \min \{ \alpha\, \ell^{n-1} \exp(-\sigma\, \ell\, t) |\hat{x}(0) - x(0)| \ ,\ \boxed{\delta \ell} \}
$$

- ✓ the peaking does not grow with ℓ^{n-1} but only with ℓ
- ✓ the first $n-1$ estimates $\hat{x}_i := \xi_{i1}$ are peaking-free
- ✓ the last estimate $\hat{x}_n := \xi_{(n-1)2}$ and the extra $n-2$ variables peak with ℓ,

The parameters (a_i, b_i) must be chosen such that **each sub – block of M is Hurwitz**

$$M = \begin{pmatrix} E_1 & N & 0 & & \dots & \dots & 0 \\ Q_2 & E_2 & N & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & Q_i & E_i & N & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & Q_{n-2} & E_{n-2} & N \\ 0 & \dots & \dots & \dots & 0 & Q_{n-1} & E_{n-1} \end{pmatrix}_{(2n-2)\times(2n-2)}$$

$$E_i = \begin{pmatrix} -a_i & 1 \\ -b_i & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & a_i \\ 0 & b_i \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

✓ the eigenvalues of M can be arbitrarily assigned

The parameters (a_i, b_i) must be chosen such that **each sub – block of M is Hurwitz**

$$M = \begin{pmatrix} E_1 & N & 0 & & \cdots & \cdots & 0 \\ Q_2 & E_2 & N & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & Q_i & E_i & N & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & Q_{n-2} & E_{n-2} & N \\ 0 & \cdots & \cdots & \cdots & 0 & Q_{n-1} & E_{n-1} \end{pmatrix}_{(2n-2)\times(2n-2)}$$

$$E_i = \begin{pmatrix} -a_i & 1 \\ -b_i & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & a_i \\ 0 & b_i \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- ✓ the eigenvalues of M can be arbitrarily assigned
- ✓ we ask for extra-conditions which can be easily verified
- ✓ it is always possible to satisfy the previous conditions

1

$$\begin{aligned} \dot{\xi}_{11} &= \xi_{12} + \ell \; a_1 (y - \xi_{11}) \\ \dot{\xi}_{12} &= \xi^s_{22} + \ell^2 \, b_1 (y - \xi_{11}) \end{aligned}$$

$\hat{x}_1 := \xi_{11}$

i

$$\begin{aligned} \dot{\xi}_{i1} &= \xi_{i2} + \ell \; a_i (\xi^s_{(i-1)2} - \xi_{i1}) \\ \dot{\xi}_{i2} &= \xi^s_{(i+1)2} + \ell^2 \, b_i (\xi^s_{(i-1)2} - \xi_{i1}) \end{aligned}$$

$\hat{x}_i := \xi_{i1}$

$n-1$

$$\begin{aligned} \dot{\xi}_{(n-1)1} &= \xi_{(n-1)2} + \ell \; a_{n-1} (\xi^s_{(n-2)2} - \xi_{(n-1)1}) \\ \dot{\xi}_{(n-1)2} &= \varphi_s(\hat{x}) + \ell^2 \, b_{n-1} (\xi^s_{(n-2)2} - \xi_{(n-1)1}) \end{aligned}$$

$\hat{x}_{n-1} := \xi_{(n-1)1}$

Removing peaking from the last estimate

1

$$\begin{aligned} \dot\xi_{11} &= \xi_{12} + \ell \; a_1 (y - \xi_{11}) \\ \dot\xi_{12} &= \xi^s_{22} + \ell^2 \, b_1 (y - \xi_{11}) \end{aligned} \quad \longrightarrow \quad \hat x_1 := \xi_{11}$$

$\updownarrow$

i

$$\begin{aligned} \dot\xi_{i1} &= \xi_{i2} + \ell \; a_i (\xi^s_{(i-1)2} - \xi_{i1}) \\ \dot\xi_{i2} &= \xi^s_{(i+1)2} + \ell^2 \, b_i (\xi^s_{(i-1)2} - \xi_{i1}) \end{aligned} \quad \longrightarrow \quad \hat x_i := \xi_{i1}$$

$\updownarrow$

$n-1$

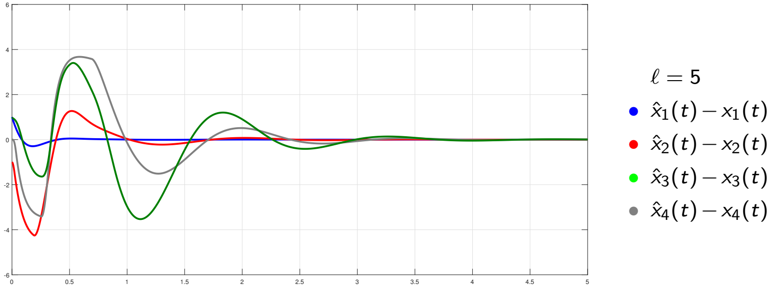
$$\begin{aligned} \dot\xi_{(n-1)1} &= \xi_{(n-1)2} + \ell \; a_{n-1} (\xi^s_{(n-2)2} - \xi_{(n-1)1}) \\ \dot\xi_{(n-1)2} &= \varphi_s(\hat x) + \ell^2 \, b_{n-1} (\xi^s_{(n-2)2} - \xi_{(n-1)1}) \end{aligned} \quad \longrightarrow \quad \hat x_{n-1} := \xi_{(n-1)1}$$

$\updownarrow$

n

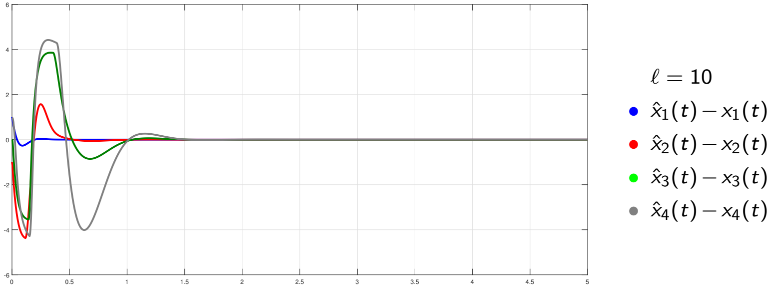
$$\dot\xi_{n1} = \varphi_s(\hat x) + \ell \, a_n (\xi^s_{(n-1)2} - \xi_{n1}) \quad \longrightarrow \quad \hat x_n := \xi_{n1}$$

- Dimension of the system: 4
- Dimension of the observer: 7

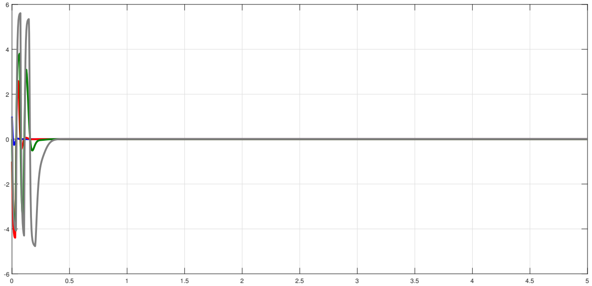


Peaking-free Low-Power Design: an example

- Dimension of the system: 4
- Dimension of the observer: 7



- Dimension of the system: 4
- Dimension of the observer: 7



$\ell = 50$

- $\hat{x}_1(t) - x_1(t)$
- $\hat{x}_2(t) - x_2(t)$
- $\hat{x}_3(t) - x_3(t)$
- $\hat{x}_4(t) - x_4(t)$

✓ We can apply the same *low-power* methodology to structures more general

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, u) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n, u) \end{aligned} \qquad y = h(x_1, u)$$

✗ The new observer has dimension $2n - 1$ and it embeds only powers ℓ, ℓ^2

✓ The poles of the observer can be arbitrarily assigned

✓ The proposed observer guarantees the **same properties** of a standard high-gain observer (asymptotic estimate, tunability property, robustness to uncertainties, ISS w.r.t. measurement noise)

✓ The estimates $\hat{x}_1, \dots, \hat{x}_n$ do not have peaking (remain in a compact set $X_\epsilon \supset X$)

✓ **Numerical issue** is **not** any more a problem: implementation of parameters in ℓ, ℓ^2 and variables belonging to $O(\ell)$ for a system of any dimension n

Peaking-free Low-Power Design: summary

✓ We can apply the same *low-power* methodology to structures more general

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, u) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n, u) \end{aligned} \qquad y = h(x_1, u)$$

✗ The new observer has dimension $2n - 1$ and it embeds only powers ℓ, ℓ^2

✓ The poles of the observer can be arbitrarily assigned

✓ The proposed observer guarantees the **same properties** of a standard high-gain observer (asymptotic estimate, tunability property, robustness to uncertainties, ISS w.r.t. measurement noise)

✓ The estimates $\hat{x}_1, \dots, \hat{x}_n$ do not have peaking (remain in a compact set $X_\epsilon \supset X$)

✓ **Numerical issue** is **not** any more a problem: implementation of parameters in ℓ, ℓ^2 and variables belonging to $O(\ell)$ for a system of any dimension n

Peaking-free Low-Power Design: summary

✓ We can apply the same *low-power* methodology to structures more general

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, u) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n, u) \end{aligned} \qquad y = h(x_1, u)$$

✗ The new observer has dimension $2n - 1$ and it embeds only powers ℓ, ℓ^2

✓ The poles of the observer can be arbitrarily assigned

✓ The proposed observer guarantees the **same properties** of a standard high-gain observer (asymptotic estimate, tunability property, robustness to uncertainties, ISS w.r.t. measurement noise)

✓ The estimates $\hat{x}_1, \ldots, \hat{x}_n$ do not have peaking (remain in a compact set $X_\epsilon \supset X$)

✓ **Numerical issue** is **not** any more a problem: implementation of parameters in ℓ, ℓ^2 and variables belonging to $O(\ell)$ for a system of any dimension n

Nonlinear observers:

Sensitivity to

high-frequency measurement noise

High-gain observer: sensitivity to high-frequency measurement noise

✗ The characterization of the effect of the noise in the nonlinear framework is not an easy task (complicated by the nonlinearities)

✗ Current methods (as far as I know) give only the L^∞ gain

- L^∞ gain of the standard HGO: $\gamma\,\ell^{n-1}\|\nu\|_\infty$
- L^∞ gain of the Low-Power HGO: $\gamma\,\ell^{n-1}\|\nu\|_\infty$

✗ Lypaunov Analysis fails in catching the behaviour of the observer at high-frequency (effect of relative degree)

We start studying the properties of a linear High-Gain Observer of dimension 2:

State-Space

$$\dot{\xi}_1 = \xi_2 + \ell\, k_1(y - \xi_1)$$
$$\dot{\xi}_2 = \ell^2\, k_2(y - \xi_1)$$

Transfer function

$$G(s) = \frac{1}{s^2 + k_1\ell\, s + k_2\ell^2}\begin{bmatrix} k_1\ell\, s + k_2\ell^2 \\ k_2\ell^2\, s \end{bmatrix}$$

$$\xi_1(s) = G_1(s)y(s)$$
$$\xi_2(s) = G_2(s)y(s)$$

High-gain observer: sensitivity to high-frequency measurement noise

- ✗ The characterization of the effect of the noise in the nonlinear framework is not an easy task (complicated by the nonlinearities)
- ✗ Current methods (as far as I know) give only the L^∞ gain
- L^∞ gain of the standard HGO: $\gamma\,\ell^{n-1}\|\nu\|_\infty$
- L^∞ gain of the Low-Power HGO: $\gamma\,\ell^{n-1}\|\nu\|_\infty$
- ✗ Lypaunov Analysis fails in catching the behaviour of the observer at high-frequency (effect of relative degree)

We start studying the properties of a linear High-Gain Observer of dimension 2:

State-Space

$$\dot{\xi}_1 = \xi_2 + \ell\, k_1(y - \xi_1)$$
$$\dot{\xi}_2 = \ell^2\, k_2(y - \xi_1)$$

Transfer function

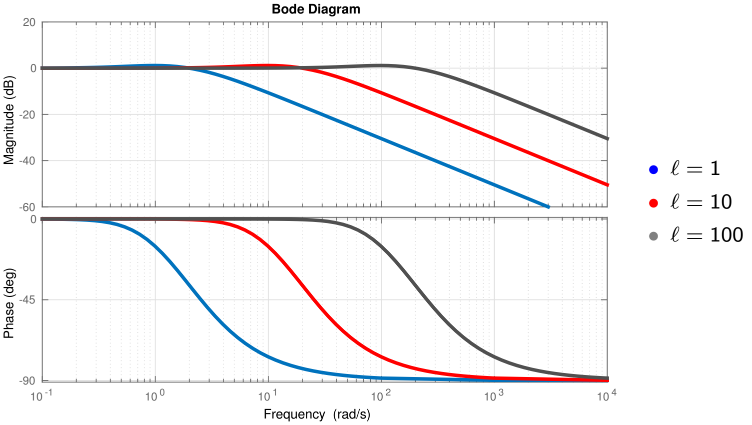
$$G(s) = \frac{1}{s^2 + k_1\ell\, s + k_2\ell^2}\begin{bmatrix} k_1\ell\, s + k_2\ell^2 \\ k_2\ell^2\, s \end{bmatrix}$$

$$\xi_1(s) = G_1(s)y(s)$$
$$\xi_2(s) = G_2(s)y(s)$$

$$G_1(j\omega) = \frac{k_1 \ell\, j\omega + k_2 \ell^2}{(j\omega)^2 + k_1 \ell\, j\omega + k_2 \ell^2}$$

$$L^\infty(G_1) = \gamma$$

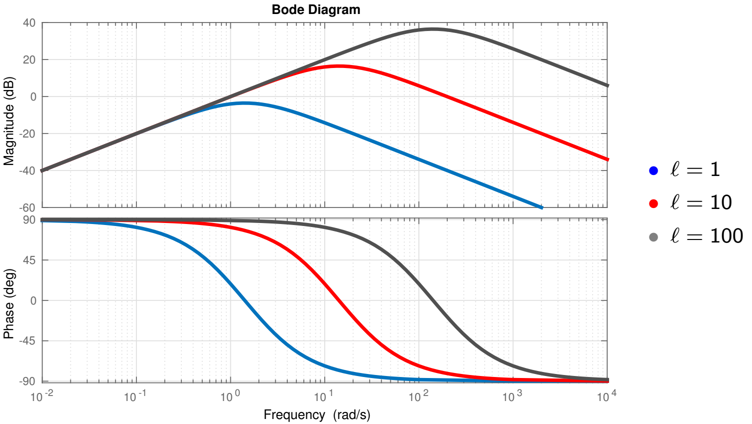
$$|G_1(j\omega)| = \beta \boxed{\frac{\ell}{\omega}} \quad \text{for } \omega \text{ large}$$



$$G_2(j\omega) = \frac{k_2 \ell^2}{(j\omega)^2 + k_1 \ell\, j\omega + k_2 \ell^2}$$

$$L^\infty(G_2) = \gamma\, \ell$$

$$|G_2(j\omega)| = \beta\, \boxed{\frac{\ell^2}{\omega}} \quad \text{for } \omega \text{ large}$$



Linear HGO

$$\left\{\begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 + \ell\, k_1(y-\xi_1) \\ \dot{\xi}_2 & = & \xi_3 + \ell^2\, k_2(y-\xi_1) \\ \dot{\xi}_3 & = & \ell^3\, k_3(y-\xi_1) \end{array}\right.$$

Linear Low-Power HGO

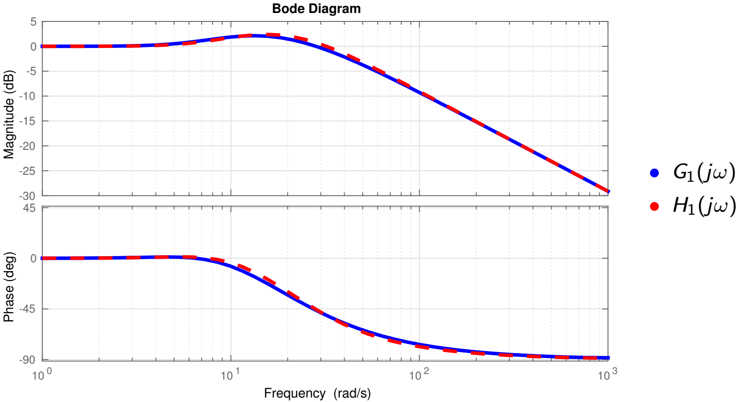
$$\left\{\begin{array}{rcl} \dot{\xi}_1 & = & \eta_1 + \ell\; a_1(y-\xi_1) \\ \dot{\eta}_1 & = & \eta_2 + \ell^2 b_1(y-\xi_1) \\ \dot{\xi}_2 & = & \eta_2 + \ell\; a_2(\eta_1-\xi_2) \\ \dot{\eta}_2 & = & \ell^2 b_2(\eta_1-\xi_2) \\ \dot{\xi}_3 & = & \ell\; a_3(\eta_2-\xi_3) \end{array}\right.$$

$$\begin{array}{rcl} \xi_1(j\omega) & = & G_1(j\omega)y(j\omega) \\ \xi_2(j\omega) & = & G_2(j\omega)y(j\omega) \\ \xi_3(j\omega) & = & G_3(j\omega)y(j\omega) \end{array}$$

$$\begin{array}{rcl} \xi_1(j\omega) & = & H_1(j\omega)y(j\omega) \\ \xi_2(j\omega) & = & H_2(j\omega)y(j\omega) \\ \xi_3(j\omega) & = & H_3(j\omega)y(j\omega) \end{array}$$

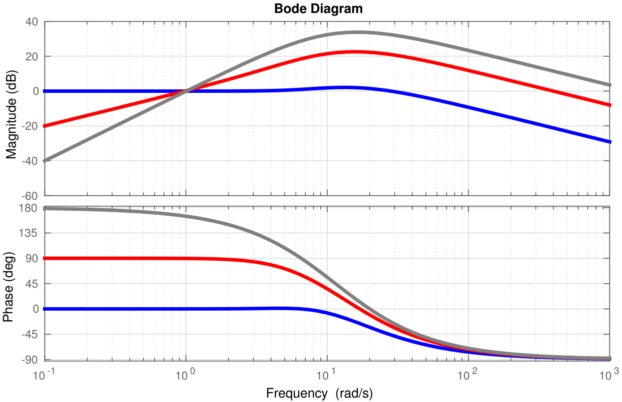
The coefficients k_1, k_2, k_3 and $a_1, a_2, a_3, b_1, b_2, b_3$ are chosen so that $G_1(j\omega)$ and $H_1(j\omega)$ have the same band.

Comparison between HGO and the Low-Power: $G_1(j\omega)$ vs $H_1(j\omega)$



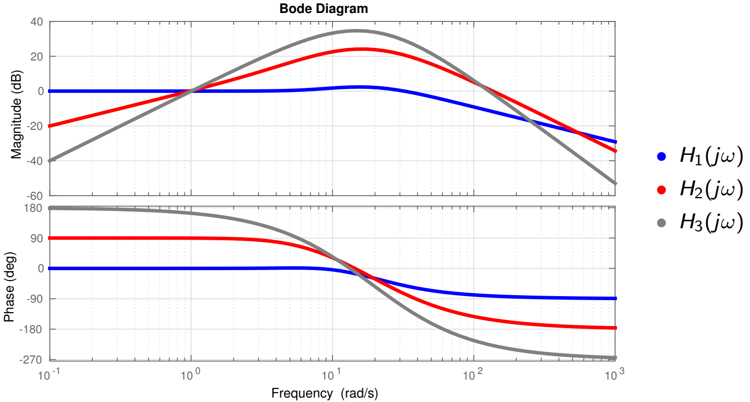
Comparison between Low-Power and standard HGO

Bode diagram of the linear HGO: $G_1(j\omega)$, $G_2(j\omega)$, $G_3(j\omega)$



- $G_1(j\omega)$
- $G_2(j\omega)$
- $G_3(j\omega)$

Bode diagram of the linear Low-Power HGO: $H_1(j\omega), H_2(j\omega), H_3(j\omega)$



- Consider a plant

$$\begin{aligned} \dot{x} &= Ax + B\varphi(x) \\ y &= Cx + \nu(t) \end{aligned}$$

and the (asymptotically convergent) observer

$$\dot{\hat{x}} = A\hat{x} + B\varphi(\hat{x}) + K(y - C\hat{x})$$

- Suppose the noise is generated by a linear system neutrally stable (oscillators) of the form

$$\varepsilon\,\dot{w} = S\,w\,, \qquad \nu = P\,w\,,$$

$$S = \mathrm{blockdiag}(S_1, \dots, S_m)\,, \qquad S_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix},$$

where ε modulate the frequency of the oscillators.

- Study the steady-state behaviour of the error system

$$\dot{e} = (A - KC)e + B\Delta_\varphi(e, x) + K\nu(t)$$

The set $\mathrm{graph}(\pi_\varepsilon) = \{((w, x, e)) \in W \times X \times \mathbb{R}^n \,:\, e \in \pi_\varepsilon(w, x)\}$ is asymptotically stable for the e-dynamics.

- We study an approximation of the solution of the PDE modelling $\pi_\varepsilon(w, x)$

Effect of high-frequency measurement noise: a novel approach

- Consider a plant

$$\begin{aligned} \dot{x} &= Ax + B\varphi(x) \\ y &= Cx + \nu(t) \end{aligned}$$

and the (asymptotically convergent) observer

$$\dot{\hat{x}} = A\hat{x} + B\varphi(\hat{x}) + K(y - C\hat{x})$$

- Suppose the noise is generated by a linear system neutrally stable (oscillators) of the form

$$\varepsilon\,\dot{w} = S\,w\,, \qquad \nu = P\,w\,,$$

$$S = \mathrm{blockdiag}(S_1, \ldots, S_m)\,, \qquad S_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix},$$

where ε modulate the frequency of the oscillators.

- Study the steady-state behaviour of the error system

$$\dot{e} = (A - KC)e + B\Delta_\varphi(e, x) + K\nu(t)$$

The set $\mathrm{graph}(\pi_\varepsilon) = \{((w, x, e)) \in W \times X \times \mathbb{R}^n \ : \ e \in \pi_\varepsilon(w, x)\}$ is asymptotically stable for the e-dynamics.

- We study an approximation of the solution of the PDE modelling $\pi_\varepsilon(w, x)$

Effect of high-frequency measurement noise: a novel approach

- Consider a plant

$$\begin{aligned} \dot{x} &= Ax + B\varphi(x) \\ y &= Cx + \nu(t) \end{aligned}$$

  and the (asymptotically convergent) observer

$$\dot{\hat{x}} = A\hat{x} + B\varphi(\hat{x}) + K(y - C\hat{x})$$

- Suppose the noise is generated by a linear system neutrally stable (oscillators) of the form

$$\varepsilon \dot{w} = S w\,, \qquad \nu = P w\,,$$

$$S = \mathrm{blockdiag}(S_1, \ldots, S_m)\,, \qquad S_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix},$$

  where ε modulate the frequency of the oscillators.

- Study the steady-state behaviour of the error system

$$\dot{e} = (A - KC)e + B\Delta_\varphi(e, x) + K\nu(t)$$

  The set $\mathrm{graph}(\pi_\varepsilon) = \{((w, x, e)) \in W \times X \times \mathbb{R}^n \,:\, e \in \pi_\varepsilon(w, x)\}$ is asymptotically stable for the e-dynamics.

- We study an approximation of the solution of the PDE modelling $\pi_\varepsilon(w, x)$

Effect of high-frequency measurement noise: a novel approach

High-Gain Observer

- high-frequency gain (for ε small enough)

$$\limsup_{t\to\infty} |\hat{x}_i(t) - x_i(t)| \leq \beta \, \boxed{\varepsilon \, \ell^i} \, \|\nu(\cdot)\|_\infty$$

Low-Power High-Gain Observer

- high-frequency gain (for ε small enough) for $i = 1, \ldots, m$ where $m = \left\lceil \frac{n+1}{2} \right\rceil$

$$\limsup_{t\to\infty} |\hat{x}_i(t) - x_i(t)| \leq \beta \, \boxed{\varepsilon^i \, \ell^{2i-1}} \, \|\nu(\cdot)\|_\infty$$

- high-frequency gain (for ε small enough) for $i = m+1, \ldots, n$

$$\limsup_{t\to\infty} |\hat{x}_i(t) - x_i(t)| \leq \beta \, \boxed{\varepsilon^{n-i+2} \, \ell} \, \|\nu(\cdot)\|_\infty$$

Remark: $\lim_{\varepsilon\to 0} \lim_{t\to\infty} |\hat{x}_i(t) - x_i(t)| = 0$

Effect of high-frequency measurement noise: a novel approach

High-Gain Observer

- high-frequency gain (for ε small enough)

$$\limsup_{t\to\infty} |\hat{x}_i(t) - x_i(t)| \le \beta \, \boxed{\varepsilon \, \ell^i} \, \|\nu(\cdot)\|_\infty$$

Low-Power High-Gain Observer

- high-frequency gain (for ε small enough) for $i = 1, \dots, m$ where $m = \left\lceil \frac{n+1}{2} \right\rceil$

$$\limsup_{t\to\infty} |\hat{x}_i(t) - x_i(t)| \le \beta \, \boxed{\varepsilon^i \, \ell^{2i-1}} \, \|\nu(\cdot)\|_\infty$$

- high-frequency gain (for ε small enough) for $i = m+1, \dots, n$

$$\limsup_{t\to\infty} |\hat{x}_i(t) - x_i(t)| \le \beta \, \boxed{\varepsilon^{n-i+2} \, \ell} \, \|\nu(\cdot)\|_\infty$$

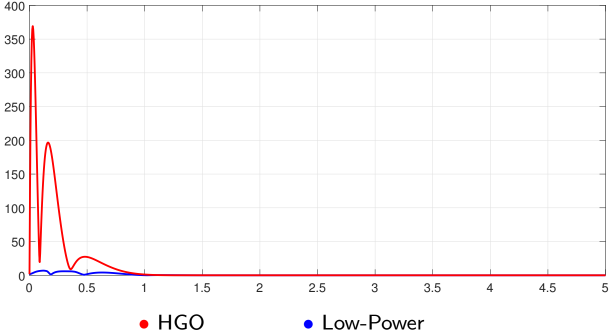
Remark: $\lim_{\varepsilon\to 0} \lim_{t\to\infty} |\hat{x}_i(t) - x_i(t)| = 0$

Effect of the measurement noise: a simulative example

- System dimension $n = 4$
- High Gain Observer dimension $n = 4$
- Low-Power High-Gain Observer dimension $2n - 1 = 7$
- Gains and coefficients chosen to get (approximatively) the same rate of convergence for both observers
- Poles of the rescaled-closed-loop matrix for standard high-gain observer in the range $[-4, -1]$ and $\ell = 6$
- Poles of the rescaled-closed-loop matrix for low-power high-gain observer in the range $[-4, -1]$ and $\ell = 10$
- High-frequency measurement noise (coloured white noise) of amplitude ≤ 0.2

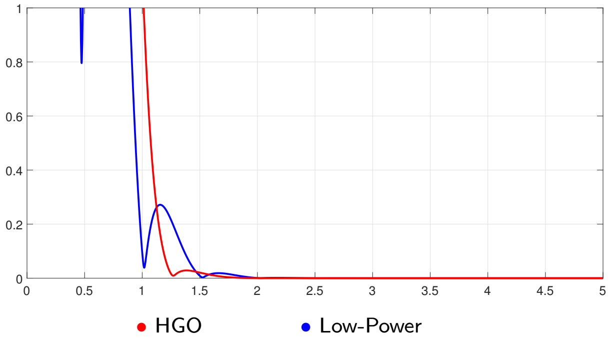
- System dimension $n = 4$
- High Gain Observer dimension $n = 4$
- Low-Power High-Gain Observer dimension $2n - 1 = 7$
- Gains and coefficients chosen to get (approximatively) the same rate of convergence for both observers
- High-frequency measurement noise (coloured white noise) of amplitude ≤ 0.2

Nominal convergence (no measurement noise): $|\hat{x}(t) - x(t)|$



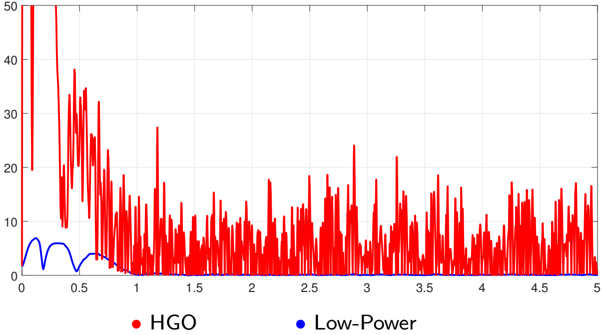
- System dimension $n = 4$
- High Gain Observer dimension $n = 4$
- Low-Power High-Gain Observer dimension $2n - 1 = 7$
- Gains and coefficients chosen to get (approximatively) the same rate of convergence for both observers
- High-frequency measurement noise (coloured white noise) of amplitude ≤ 0.2

Nominal convergence (no measurement noise) zoom: $|\hat{x}(t) - x(t)|$



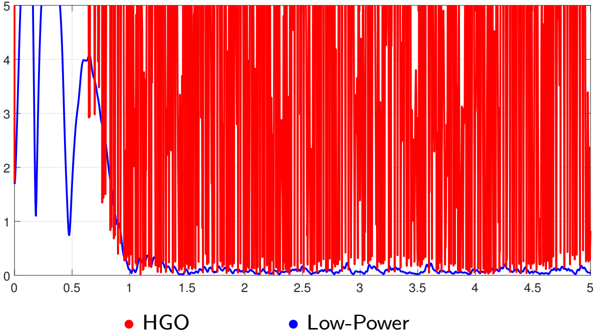
- System dimension $n = 4$
- High Gain Observer dimension $n = 4$
- Low-Power High-Gain Observer dimension $2n - 1 = 7$
- Gains and coefficients chosen to get (approximatively) the same rate of convergence for both observers
- High-frequency measurement noise (coloured white noise) of amplitude ≤ 0.2

Convergence in presence of noise: $|\hat{x}(t) - x(t)|$



- System dimension $n = 4$
- High Gain Observer dimension $n = 4$
- Low-Power High-Gain Observer dimension $2n - 1 = 7$
- Gains and coefficients chosen to get (approximatively) the same rate of convergence for both observers
- High-frequency measurement noise (coloured white noise) of amplitude ≤ 0.2

Convergence in presence of noise zoom: $|\hat{x}(t) - x(t)|$



References on low-power high-gain observers

1) Astolfi D. and Marconi L., "*A high-gain nonlinear observer with limited gain power*", IEEE Transactions on Automatic Control, Vol. 60 (11), pp. 3059-3064, November 2015

2) L. Wang, D. Astolfi, H. Su, L. Marconi, "*High-gain observers with limited gain power for systems with observability canonical form*", to appear on Automatica, 2017

3) D. Astolfi, L. Marconi, L. Praly and A. Teel, "*Low-power peaking-free high-gain observers*", submitted to Automatica

4) L. Wang, D. Astolfi, H. Su, L. Marconi, A. Isidori, "*Output stabilization for a class of nonlinear systems via high-gain observer with limited gain power*", presented at MICNON 2015, Volume 48 (11), pp. 730-735, June 2015

5) D. Astolfi, L. Marconi, A. Teel, "*Low-power peaking-free high-gain observers for nonlinear systems*", presented at ECC 2016

6) D. Astolfi, L. Marconi, L. Praly, A. Teel, "*Sensitivity to High-Frequency Measurement Noise of Nonlinear High-Gain Observers*", presented at NOLCOS 2016