# Synthesis of infinite-dimensional observers for a class of vibrating systems : rotating body-beam example

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# Outlines

1. **Luenberger observer : from finite to infinite-dim. space**
2. **Exact observability and exponential convergence**
3. **Application to a rotating body-beam system**
4. **Conclusions**

# Finite-dimensional Luenberger-like observer

harmonic oscillator

$$\begin{cases} \dot{x} = Ax, \quad x(0) = x_0 \\ y = Cx \end{cases}$$

with $A^T = -A$, $(A, C)$ observable.

Luenberger-like observer

$$\dot{\hat{x}} = A\hat{x} - \kappa C^T(C\hat{x} - y), \quad \hat{x}(0) = \hat{x}_0$$

with $\kappa > 0$.

error system

$$\dot{\varepsilon} = (A - \kappa C^T C)\varepsilon, \quad \varepsilon(0) = \varepsilon_0$$

with $\varepsilon = \hat{x} - x$.

convergence

LaSalle's principle $\Rightarrow$ *asymptotic* stability ;

$\hat{x}(t) \to x(t)$ *exponentially* <u>for any $\kappa > 0$</u>

# Infinite-dim. observer with unbounded observation

linear system in Hilbert space

$$\begin{cases} \dot{w}(t) = Aw(t) + Bu(t), \\ w(0) = w_0, \\ y(t) = Cw(t) \end{cases}$$

with $w \in X$, $u \in U$, $y \in Y$;

$A$ the generator of a $C_0$ unitary group $T_t := e^{At}$ on $X$;

$B \in \mathcal{L}(U; X_{-1})$ with $X_{-1} = \bar{X}$, $\|z\|_{-1} = \|(\beta I - A)^{-1} z\|$, $\beta \in \rho(A)$;

$C \in \mathcal{L}(X_1; Y)$, with $X_1 = \mathcal{D}(A)$, $\|z\|_1 = \|(\beta I - A) z\|$

$C$ unbounded (*example :* $Cf = \partial_x f(0)$)

Luenberger-like observer *(formally)*

$$\begin{cases} \dot{\hat{w}}(t) = A\hat{w}(t) - \kappa\, C^*[C\hat{w}(t) - y(t)] + Bu(t), \quad \kappa > 0, \\ \hat{w}(0) = \hat{w}_0, \end{cases}$$

error system

$$\dot{\varepsilon} = (A - \kappa C^T C)\varepsilon := A^\kappa \varepsilon, \quad \varepsilon(0) = \varepsilon_0$$

# Exact observability and regularity

A couple $(A, B)$ is *admissible* if, $\forall \tau > 0$,

$$\int_0^\tau T_{\tau - t} B u(t)\, dt \in X, \quad \forall\, u \in L^2_{loc}(\mathbb{R}^+; U).$$

A couple $(A, C)$ is *admissible* if, $\forall \tau > 0$, $\exists K_\tau > 0$ s.t.

$$\int_0^\tau \| C T_t w_0 \|_Y^2\, dt \leq K_\tau \, \|w_0\|_X^2, \quad \forall\, \underline{w_0 \in \mathcal{D}(A)}.$$

A couple $(A, C)$ is *exactly observable* over $\tau$ if it is admissible and there exists some positive constants $\tau$ and $k_\tau$ such that

$$k_\tau \|w_0\|_X^2 \leq \int_0^\tau \| C T_t w_0 \|_Y^2\, dt, \quad \forall\, w_0 \in \mathcal{D}(A).$$

$\Rightarrow$ We can determine exactly the *state* of the system from the observation of its *input* and *output* on a sufficiently long time interval.

# Exact observability and regularity

The linear system is *regular* if

$i)$ $(A, B)$ and $(A, C)$ are admissible ;

$ii)$ the system is *well-posed* s. t. there exists a $D \in \mathcal{L}(U; Y)$,

$$\lim_{s \to +\infty} G(s)u := \lim_{s \to +\infty} Du + C_\Lambda (sI - A)^{-1} Bu = Du, \quad \forall\, u \in U,$$

· $G(s)$ the transfer function which is analytic and bounded in the right-half complex plane $\mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \Re(s) > \alpha\}$ for some $\alpha \in \mathbb{R}^+$ ;

· $C_\Lambda$ is the $\Lambda$-*extension* of $C$ defined by

$$C_\Lambda w_0 := \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} w_0, \quad \forall w_0 \in \mathcal{D}(C_\Lambda),$$

with $\mathcal{D}(C_\Lambda)$ consists of <u>all $x \in X$ for which the limit exists</u>.

$\Rightarrow$ Regularity : $C_\Lambda(sI - A)^{-1}B$ makes sense, for every $s \in \rho(A)$.

# Exponential convergence of the observer

For the infinite-dimensional error system, a large gain of correction $\kappa$ may provoke instability of the error system

## Theorem

Let $A$ be the generator of a $C_0$ unitary group on $X$ and let both $(A, C^*, C)$ and $(A, B, C)$ be *regular* s. t. $(A, C)$ is *exactly observable*. Then there exist some positive constants $K_{\min}$ and $K_{\max}$ with $0 \leq K_{\min} \leq K_{\max} < \infty$ s. t.

- the observer **converges** *exponentially* if $0 < \kappa < 1/K_{\max}$ ;
- the observer **diverges** *exponentially* if $\kappa > 1/K_{\min}$ is an admissible feedback for $(A, C^*, C)$.

$$K_{\max} = \sup_{\substack{f \in \mathrm{Ran}(C_\Lambda) \\ |f|=1}} \lim_{\substack{\beta \in \mathbb{R}^+ \\ \beta \to +\infty}} \beta \|(\beta - A)^{-1} C^* f\|_X^2,$$

$$K_{\min} = \inf_{\substack{f \in \mathrm{Ran}(C_\Lambda) \\ |f|=1}} \lim_{\substack{\beta \in \mathbb{R}^+ \\ \beta \to +\infty}} \beta \|(\beta - A)^{-1} C^* f\|_X^2.$$

*example*

transport equation

$$\begin{cases} w_t(x,t) = w_x(x,t), \\ w(0,t) = w(1,t), \\ y(t) = w(0,t) \end{cases}$$

state space $X = L^2(0,1)$, observation space $Y = \mathbb{R}$.

Luenberger observer

$$\begin{cases} \hat{w}_t(x,t) = \hat{w}_x(x,t), \\ \hat{w}(0,t) = \hat{w}(1,t) + \kappa[\hat{w}(0,t) - y(t)]. \end{cases}$$

- The example has $K_{\max} = K_{\min} = 1/2$ : the observer is
  *exponentially* **convergent** for $0 < \kappa < 2$
  *exponentially* **divergent** for $\kappa > 2$.
- If the dimension of state space is *finite*, we will have $K_{\max} = K_{\min} = 0$.

# Exponential convergence of the observer

## Lemma

$$\langle A^\kappa \varepsilon, \varepsilon\rangle_X \leq -\kappa(1-\kappa K_{\max})|C_\Lambda \varepsilon|_Y^2, \qquad \forall \varepsilon \in \mathcal{D}(A^\kappa),\ \forall 0<\kappa<1/K_{\max}$$
$$\langle A^\kappa \varepsilon, \varepsilon\rangle_X \geq \kappa(\kappa K_{\min}-1)|C_\Lambda \varepsilon|_Y^2, \qquad \forall \varepsilon \in \mathcal{D}(A^\kappa),\ \forall \kappa>1/K_{\min}.$$

- $0<\kappa<1/K_{\max}$

$$\frac{d|\varepsilon|_X^2}{dt} = 2\langle A^\kappa \varepsilon, \varepsilon\rangle_X \quad \Rightarrow \quad |\varepsilon(T)|_X^2 \leq |\varepsilon_0|_X^2 - 2\kappa(1-\kappa K_{\max})\int_0^T |C_\Lambda \varepsilon(\tau)|_Y^2\, d\tau$$
$$\text{exact observability} \quad \Rightarrow \quad \int_0^T |C_\Lambda \varepsilon(t)|_Y^2\, dt \geq K|\varepsilon_0|_X^2$$

$$\Rightarrow |\varepsilon(T)|_X^2 \leq [1-2\kappa(1-\kappa K_{\max})K]|\varepsilon_0|_X^2 \quad \Rightarrow |\varepsilon(t)|_X^2 \leq M e^{-rt}|\varepsilon_0|_X^2$$

$$M = \sup_{t\in[0,T]} |e^{tA^\kappa}|^2,\ r = T^{-1}\ln[(1-2\kappa(1-\kappa K_{\max})K)^{-1}]$$

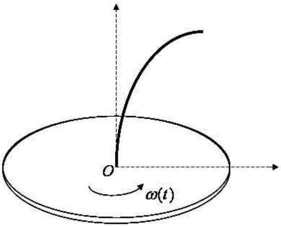
- $\kappa > 1/K_{\min}$

$$|\varepsilon(T)|_X^2 \geq |\varepsilon_0|_X^2 + 2\kappa(\kappa K_{\min}-1)\int_0^T |C_\Lambda \varepsilon(\tau)|_Y^2\, d\tau$$

$$\Rightarrow |\varepsilon(T)|_X^2 \geq [1+2\kappa(\kappa K_{\min}-1)K]|\varepsilon_0|_X^2 \quad \Rightarrow |\varepsilon(t)|_X^2 \geq \bar{M} e^{-\bar{r}t}|\varepsilon_0|_X^2$$

simplified model

- rotating disc
- flexible Euler-Bernoulli beam




stabilization + observability :

interesting info for control of mechanical systems

(*example : satellites with solar panels, robot with flexible joint...*)

# Rotating body-beam system

hybrid system governed by PDE and ODE

- Lagrangian formulation $\Rightarrow$

$$\begin{cases} \rho w_{tt}(x,t) + EI\, w_{xxxx}(x,t) + \rho B w_t(x,t) = \rho \omega^2(t) w(x,t), \\ \Gamma(t) = \dfrac{d}{dt}[\omega(t)(I_d + \int_0^L w^2(x,t)dx)] \quad \textit{(Newton's law)} \end{cases}$$

- BC : $w(0,t) = w_x(0,t) = w_{xx}(L,t) = w_{xxx}(L,t) = 0$

*$w(x,t)$ = deformation of the beam*
*$\Gamma(t)$ = torque control*
*$B$ = friction operator*
*$L$ = length of the beam $[m]$*
*$\rho$ = mass density of the beam $[kg/m]$*
*$E$ = elastic module $[N/m^2]$, $I$ = moment of inertia $[m^4]$*
*$I_d$ = moment of inertia of the disc $[kg \cdot m^2]$*
*$\omega(t)$ = angular velocity of the disc $[rad/s]$*

stabilization : find $\Gamma(t)$ s.t. $(w(\cdot,t), \omega(t)) \xrightarrow[t\to\infty]{} (0, \omega_*)$ ?

- with friction ($B \neq 0$)
- without friction ($B = 0$) + boundary control
- without boundary control
  - · damping feedback law [Jurdjevic & Quinn]
  - · backstepping + desingularisation (non $C^1$) [Praly & Coron]

state feedback control law [Coron & d'Andéa-Novel 1998]

$$\begin{aligned}\dot{\omega} = \gamma &= \mathcal{F}(\omega, w, w_t) \\ &= -\Big(\omega + \omega_* - \sigma\big(\int_0^1 w w_t \, dx\big)\Big) \int_0^1 w w_t \, dx - C_2 \Big(\omega - \omega_*\Big) \\ &\quad -\sigma'\Big(\int_0^1 w w_t\Big) \int_0^1 \Big(w_t^2 - w_{xx}^2 + \omega^2 w^2\Big) \, dx\end{aligned}$$

with $C_2 > 0$ and $\sigma \in \mathcal{C}^1(\mathbb{R}) : (2\omega_* - \sigma(s))s\sigma(s) > 0, \forall \, s \in \mathbb{R}$.

The closed-loop system is *strongly GAS* w.r.t. $(0, \omega_*)$. (with $0 < \omega_* < \omega_{crit}$)

essential points of Coron & d'Andréa's control law

the state space is of *infinite* dimension

- torque control applied on the rigid body

  $\Rightarrow$ easy to be realized in practice

- need to know the state variable which is of *infinite* dimension, while the observation space is of *finite* dimension

  $\Rightarrow$ no direct access to the state variable ($w(x,t)$, $\forall x \in [0,1]$)

- solution : estimate <u>online</u> the state variable through an observer

$$\hat{w}(x,t) \xrightarrow[t\to\infty]{} w(x,t), \forall\, \hat{w}_0$$

$\Rightarrow$ replace $\Gamma(w, w_t, \omega)$ by $\Gamma(\hat{w}, \hat{w}_t, \omega)$

# Observer and convergence

2 approaches to deal with distributed parameters systems :

- **direct method**

  discretization → observer design, control law synthesis, ... → simulation

  + : finite-dim. control theory toolbox

  - : 
    - depend on the numerical schemes [Krstic 2008]
    - loss of intrinsic properties of SPD (no convergence guaranteed)

- **indirect method**

  observer design, control law synthesis, ... → discretization → simulation

  + : conservation of properties of SPD

  - : more technical difficulties

# Observer and convergence

simplified model

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = \omega_*^2 w(x,t), & \textit{(second order)} \\ w(0,t) = w_x(0,t) = 0, \\ w_{xx}(1,t) = w_{xxx}(1,t) = 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \\ y(t) = w_{xx}(0,t). & \textit{(single boundary measurement)} \end{cases}$$

state space $X = H_L^2 \times L^2(0,1)$, $H_L^2 = \{f \in H^2(0,1); f(0) = f_x(0) = 0\}$

$$\langle f, g \rangle_X = \int_0^1 f_{1xx} g_{1xx} + f_2 g_2 - \omega_*^2 f_1 g_1 \ dx$$

- second order observer [Demetriou 2004]

  + : $\widehat{w_t} = \hat{w}_t$ ;

  - : no exp. convergence ; with friction ; measurement on velocity

- multi-measurements [Nyguen 2003]

  + : exp. convergence ;

  - : cost

Luenberger-like observer : *(first order, explicit)*

$$
\begin{cases}
\hat{w}_{1t}(x,t) = \hat{w}_2(x,t) - \kappa F(x)\{\hat{w}_{1xx}(0,t) - y(t)\}, \\
\hat{w}_{2t}(x,t) = -\hat{w}_{1xxxx}(x,t) + \omega_*^2 \hat{w}_1(x,t), \\
\hat{w}_1(0,t) = \hat{w}_{1x}(0,t) = 0, \\
\hat{w}_{1xx}(1,t) = \hat{w}_{1xxx}(1,t) = 0, \\
\hat{w}(x,0) = \hat{w}_0(x),\ \hat{w}_t(x,0) = \hat{w}_1(x)
\end{cases}
$$

with $\kappa > 0$ and $F$ the unique solution of

$$
\begin{cases}
F^{''''}(x) - \omega_*^2 F(x) = 0, \\
F(0) = F^{''}(1) = F^{'''}(1) = 0, \\
F^{'}(0) = 1.
\end{cases}
$$

$$\mathcal{D}(A^\kappa) = \{(f_1\ f_2)^T \in (H^4(0,1) \cap H^2_L) \times H^2(0,1);\ f_{2x}(0) = \kappa f_{1xx}(0),\ f_2(0) = f_{1xx}(1) = f_{1xxx}(1) = 0\}$$

$$A^\kappa \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\kappa F(x)\Psi & I \\ -\partial_x^4 + \omega_*^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A^\kappa)$$

with $\Psi f = f_{xx}(0)$.

error system

$$\begin{cases} \dot{\varepsilon}(t) = A^\kappa \varepsilon(t), \\ \varepsilon(0) = \varepsilon_0. \end{cases}$$

- *Lyapunov* stability : if $|\omega_*| < \omega_{crit}$
- *exponential* stability : ?

exponential convergence

**Theorem**

Suppose that $|\omega_*| < \omega_{crit}$. The observer is *exponentially convergent* for *every positive gain* of correction $\kappa > 0$.

Given $\kappa > 0$, there exist $M > 0$, $\alpha > 0$ s.t.

$$\left\| \begin{pmatrix} \hat{w}_1(\cdot,t) \\ \hat{w}_2(\cdot,t) \end{pmatrix} - \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \right\|_X \leq Me^{-\alpha t} \left\| \begin{pmatrix} \hat{w}_0 \\ \hat{w}_1 \end{pmatrix} - \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right\|_X$$

$$\|\varepsilon(t)\| \leq Me^{-\alpha t}\|\varepsilon_0\| \ ?$$

If $A^\kappa = A - \kappa C_\Lambda^* C$ is of compact resolvent

$$\Rightarrow \alpha = \sup_{\lambda \in \sigma_p(A^\kappa)} Re(\lambda)$$

*Proof* :

- $\forall \kappa \geq 0$, $A^\kappa$ is dissipative and the generator of $C_0$ semi-group of contractions on $X$ : Lumer-Phillips
- exact observability of the observation system $(A, C)$ on $X$

### Lemma

$\forall \omega_* \in \mathbb{R}$, the observation system is *exactly observable* with $y = w_{1xx}(0,t)$, i.e., $\exists T > 0, K > 0$ s. t.

$$K\|w_0\|_X^2 \leq \int_0^T w_{1xx}(0,t)\,dt \leq K^{-1}\|w_0\|_X^2 \qquad \forall\; w_0 \in X.$$

- exact observability of the error system $(A^\kappa, C)$

$$\Rightarrow\; V = \|\varepsilon(t)\|^2 = \|\varepsilon_0\|^2 - 2\kappa \int_0^t |C\varepsilon|^2 \leq \|\varepsilon_0\|^2 - 2\kappa K \|\varepsilon_0\|^2$$

$\Rightarrow$ exponential decay rate $r = -\ln(1 - 2\kappa K)\,T^{-1}$

exact observability of the observation system $(A, C)$ on $X$

*Proof* :

- spectral decomposition $\Rightarrow X = H_1^{m-1} \oplus H_m^\infty$
  $H_1^{m-1}$ = space generated by the first $2(m-1)$ eigenvectors de $A$
  $m$ = the smallest integer s. t. $\omega_*^2 < l_m$ with $l_m \in \sigma(\partial_x^4)$
- exact observability $(A_{|H_1^{m-1}}, CP_m)$ on $H_1^{m-1}$ : Kalman criteria
  $P_m$ = projection of $X$ onto $H_1^{m-1}$ along $H_m^\infty$
- exact observability $(A_{|H_m^\infty}, C(1-P_m))$ on $H_m^\infty$ : multiplier approach
  multiplier $(x-1)\,W_x$
- exact observability of $(A, C)$ on $X$ : simultaneous exact observability
  $\sigma(A)$ point spectrum and $\sigma(A_m) \cap \sigma(A_\infty) = \emptyset$

# Decay rate assignment

**Theorem [Guo 2002]**

The eigenvalues $(\lambda_n)_{n\geq 0}$ of the operator $A^\kappa$ are eventually with algebraic multiplicity 1 and the real part tends to $-2\kappa$ when $n \to \infty$. The sequence of the corresponding generalized eigenfunctions forms a Riesz basis on $X$. Moreover the exponential decay rate of the $C_0$-semigroup is determined by the spectre of its generator.

Accelerate the convergence rate of the observer

observer error system exactly observable

$\Rightarrow$ pole placement or spectrum assignment

$\Rightarrow$ second step in observer design : assign the convergence rate to a fixed value

# Decay rate assignment

## Corollary

On replacing $\kappa[F(x)\ 0]^T$ by $\kappa[F(x)\ 0]^T + B(x)$ with some appropriate $\kappa > 0$ and $B(x)$, the convergence rate of the observer is tuned to as quick as we like.

*example*

1 eigenvalue :

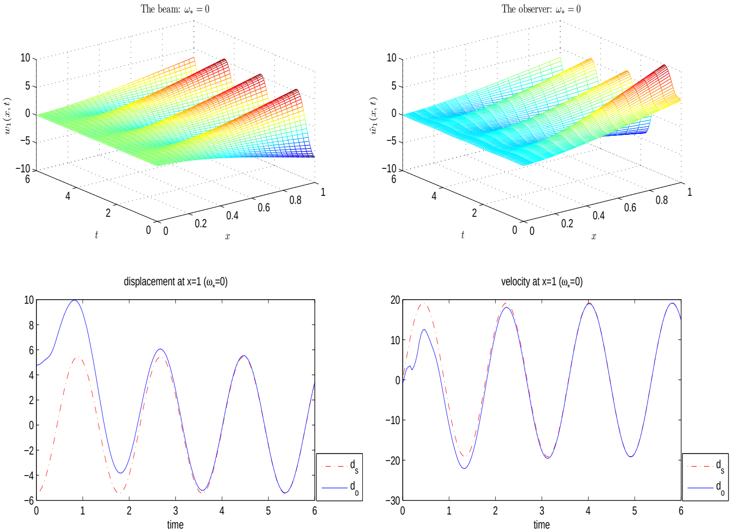
$$B = \frac{\tilde{\lambda}_1 - \lambda_1}{u_{1xx}(0)} e_1$$

allows to assign $\lambda_1$ to $\tilde{\lambda}_1$

with $e_1 = (u_1\ v_1)^T$ the eigenvector associated to $\lambda_1$.

# Simulation

$\omega = 3$ [rad/s]



# Simulation

$\omega(t) = 3 \cdot sin^2(t)$ [rad/s]

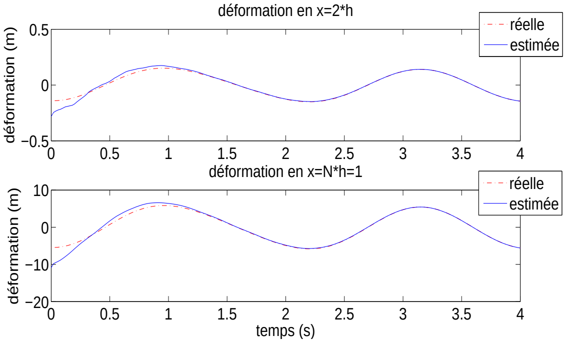



FIGURE: deformation history at $x = 0.1L$ (upper) and $x = L$ (lower) for the observation system (dashed) and the observer (solid). $\kappa = 1$.

# Simulation

## robustness

perturbation : $\tilde{y}(t) = w_{1xx}(0,t) + b(t),\ b =$ white noise
(amplitude = 20% w.r.t the measurement, mean=0)

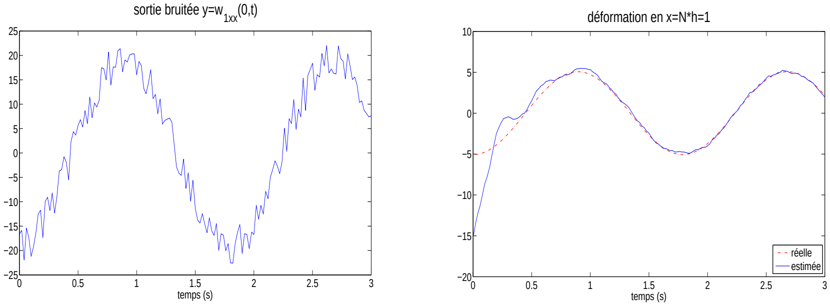


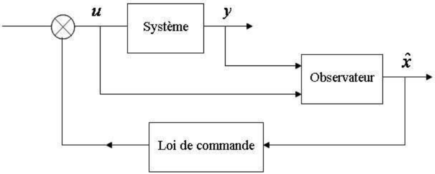

FIGURE: noise corrupted measurement (left) and deformation history (right) at $x = L$ of the observation system (dashed) and the observer (solid). $\omega_* = 0$ $[rad/s]$. $\kappa = 1$.

# Closed-loop system

## Observer-based control

- non-linear hybrid system *(PDE-ODE)*
- unbounded observation operator
- located control

principle of separation $\Gamma(w, w_t, \omega) \Rightarrow \Gamma(\hat{w}, \hat{w}_t, \omega)$




### Conjecture

Let $\bar{\omega} \in ]-\sqrt{l_1}, \sqrt{l_1}[\backslash\{0\}$. $(0, 0, \bar{\omega})$ (*resp.* $(0, 0, 0)$) is an equilibrium *locally asymptotically* stable on $X \times X \times \mathbb{R}$ for closed-loop system coupled with the feedback $\gamma$ (*resp.* $\tilde{\gamma}$).

# Conclusions

- **Summary**
  - exponentially convergent Luenberger-like observer (*proofs*)
  - application to a rotating body-beam system
  - decay rate assignment
- **Follow up works**
  - global stabilization (*proofs*)
    (principle of separation [Gauthier & Kupka 1992])

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*Thank you for your attention*