On a differential algebraic and numerical differentiation approach of observation problems

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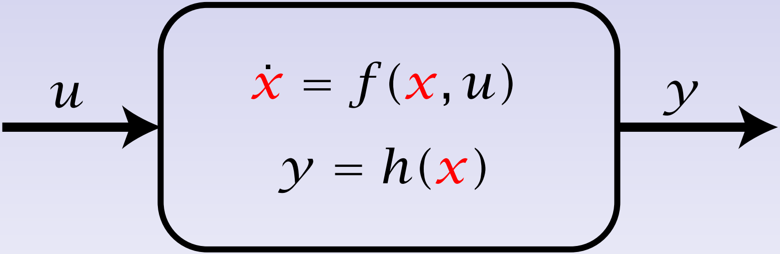
The observer design problem



Given

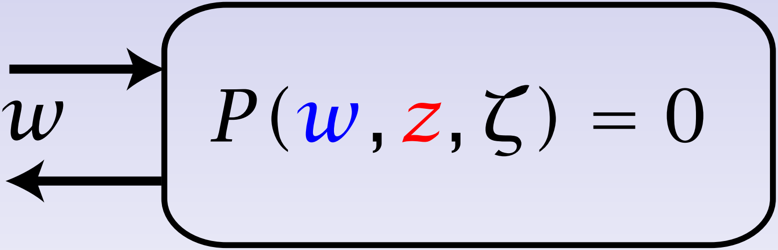
- the model, i.e., the functions f and h
- and the **online** data: $u(\tau)$, $y(\tau)$ for all $\tau \in [t_0, t]$,

calculate the **current** state value $x(t)$.

The question is twofold:

- Is it possible to **calculate** x from the model and the time history of u and y?
  Observability theory
- If yes, design a scheme to do the computation:
  Observer design

The observer design problem



Given

- the **model**, i.e., the function P
- and the **online data**: $w(\tau)$ for all $\tau \in [t_0, t]$,

calculate the **current** value of $z(t)$.

The anaeorobic digestion case

$$\text{1-stage model:} \quad \begin{cases} \dot{X} = \mu X - D X, \\ \dot{S} = -K_1 \mu X + D (S_{\text{in}} - S), \\ Q = K_2 \mu X. \end{cases}$$

- X is the concentration of bacteria, S is the concentration of substrate, D is the dilution rate, and K_1 and K_2 are constant parameters.
- The **specific growth rate** μ depends on X and S in a way that is not precisely known
- **It is desired to estimate the specific growth rate** μ from online measurements of biogas flow rate Q and the dilution rate D

Ill-posedness on a simpler example borrowed from [1]

$$\begin{cases} \dot{x}_1 = -a_1\, x_2\,, \\ \dot{x}_2 = x_1 - a_2\, x_2\,, \\ y = x_2\,. \end{cases}$$

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- Classical Kalman observability text:

$$\left\{\begin{array}{l} \dot{x} = F\ x + G\ u \\ y = H\,x \end{array}\right. \quad \text{with } F = \begin{pmatrix} 0 & -a_1 \\ 1 & -a_2 \end{pmatrix}\ G = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } H = (0\ 1)$$

$$\mathrm{rk}_{\mathbb{R}} \begin{pmatrix} H \\ H\,F \end{pmatrix} = \mathrm{rk}_{\mathbb{R}} \begin{pmatrix} 0 & 1 \\ 1 & -a_2 \end{pmatrix} = 2 = \text{number of components of } x$$

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- Kalman/Luenberger observer: $\dot{\widehat{x}} = F\,\widehat{x} + G\,u + L(y - H\,\widehat{x})$
- The observability condition implies the ability to find L such that the estimation error, $\widetilde{x} = x - \widehat{x}$, converges exponentially according to $\dot{\widetilde{x}} = (F - L\,H)\widetilde{x}$

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- Easy calculations show that

$$y(t) = e^{-a_2(t-t_0)} y(t_0) + \int_{t_0}^{t} e^{-a_2(t-\tau)} x_1(\tau) \mathrm{d}\tau$$

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$$x_1(t) = \dot{y}(t) + a_2\, y(t)$$

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- Or, given the differentiability of functions,

$$x_1(t) = \dot{y}(t) + a_2\, y(t)$$

- which shows that the estimation problem

$$y = K\, x$$

is ill-posed.

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- Considering the effect of **measurements uncertainties:** $\tilde{\bar{y}} = y - \bar{y}$
- The Kalman/Luenberger observer amplification of $\tilde{\bar{y}}$ is **bounded**:

$$\dot{\tilde{x}} = (F - L\,H)\tilde{x} + L\tilde{\bar{y}}$$

$$\tilde{x}(t) = e^{(F-L\,H)(t-t_0)}\tilde{x}(t_0) + \int_{t_0}^{t} e^{(F-L\,H)(t-\tau)} L\,\tilde{\bar{y}}(\tau)\mathrm{d}\tau$$

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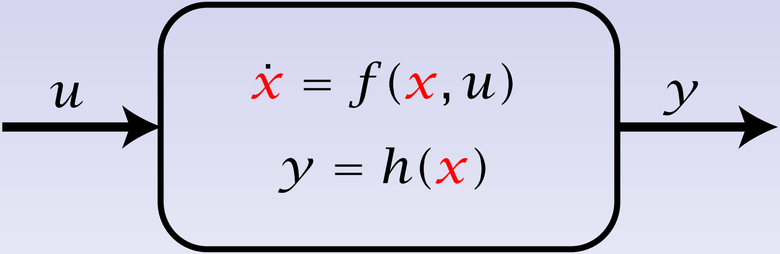
$$x = \begin{pmatrix} s + a_2 \\ 1 \end{pmatrix} y$$

- And the Laplace transform of the estimation error dynamics:

$$\widehat{x} = \begin{pmatrix} \frac{(s+a_2)\,\omega^2 - 2\,a_1\,\xi\,\omega - a_1 s}{s^2 + 2\,\xi\,\omega\, s + \omega^2} \\ \\ \frac{\omega^2 + 2\,\xi\,\omega\, s - a_2\, s - a_1}{s^2 + 2\,\xi\,\omega\, s + \omega^2} \end{pmatrix} y \quad \xrightarrow[1/\omega \longrightarrow 0]{} \widehat{x} = \begin{pmatrix} s + a_2 \\ 1 \end{pmatrix} y$$

The observation problem as an inverse problem



Assuming f to be regular enough, given $u(\tau)$ in $\tau \in [t_0, t]$ there is a map

$$K : x \mapsto y$$

calculating x means solving the inverse problem

$$K\,x = y$$

The Kalman solution

If we restrict ourselves to the linear case

$$\begin{cases} \dot{x} = F(t)x + G(t)u\,, \\ y = H(t)x\,, \end{cases} \tag{1}$$

The following dynamic system

$$\begin{cases} \dot{\widehat{x}} = F(t)x + G(t)u + L\,(y - H\,\widehat{x})\,, \\ L(t) = \dfrac{1}{2} P(t)\,H'(t)\,R(t)\,, \\ \dot{P} = F\,P + P\,F' - P\,H'R\,H\,P + Q\,, \end{cases} \tag{2}$$

verifies $\widetilde{x} = x - \widehat{x}$

$$\dot{\widetilde{x}} = (F - L\,H)\widetilde{x}$$

with exponential stability of the equilibrium point $\widetilde{x} = 0$ under some observability conditions.

The Kalman solution

- The observability conditions which guarantee the convergence are not explicitly verifiable in general
- They implicitly assume known a fundamental solution of the differential equation
$$\dot{x} = F(t)x$$
- The operator K in the observation problem is not explicitly known.

A differential algebraic approach

We consider systems of the form

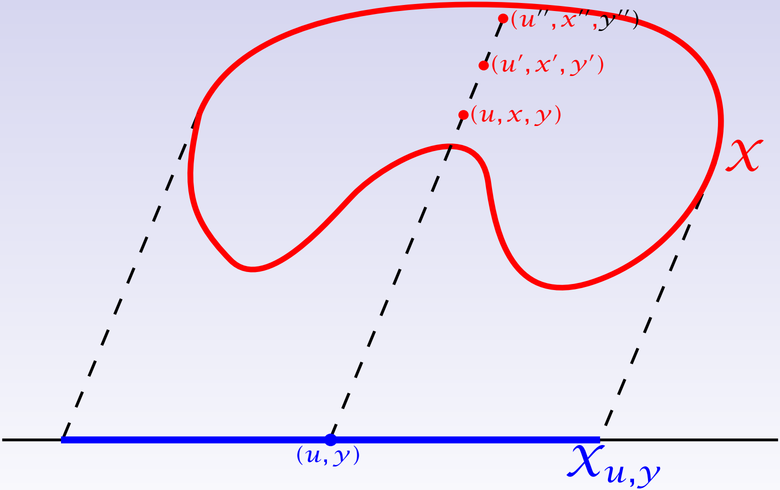
$$\begin{cases} \dot{x} = f(u, x)\,, \\ y = h(u, x)\,, \end{cases} \tag{3}$$

or, more generally,

$$\begin{cases} F_i(w, z, \zeta) = 0 \qquad i = 1, 2, \ldots \\ G(w, z, \zeta) \neq 0\,, \end{cases} \tag{4}$$

where f and h are **rational functions** and the F's and G are **differential polynomials** with coefficients in differential fields of characteristic zero.

A differential algebraic approach



- The variable z is said to be **(algebraically) observable with respect to** w if the projection map

$$\begin{array}{cccc} \pi & : & \mathcal{X} & \to & \mathcal{X}_w \\ & & (\overline{w}, \overline{z}, \overline{\zeta}) & \mapsto & \overline{w} \end{array}$$

  (sending every trajectory $(\overline{w}, \overline{z}, \overline{\zeta})$ of $\mathcal{X}$ into the corresponding observation $\overline{w}$) is **generically finite**
- If z is observable with respect to w then the degree of π is called the **observability degree** of z with respect to w, and is denoted by $\mathrm{d}_w^\circ z$
- The variable z is said to be **rationally** observable with respect to w if it is observable with respect to w with **observability degree one**
- For state systems of the form (3) we say that the **system is observable** if its **state variable is observable with respect to u and y**

x is observable iff each component is the solution of a polynomial equation

$$Hi(x_i, u, \dot{u}, ..., y, \dot{y}, ...) = 0$$

If x is rationally oberservable then the previous relation becomes

$$X_i = H_i(u, \dot{u}, ..., y, \dot{y}, ...)$$

For linear invariant systems the two approaches agree.

For linear time-varying systems (differential algebraic) observability and uniform observability **do not** agree.

Test of observability

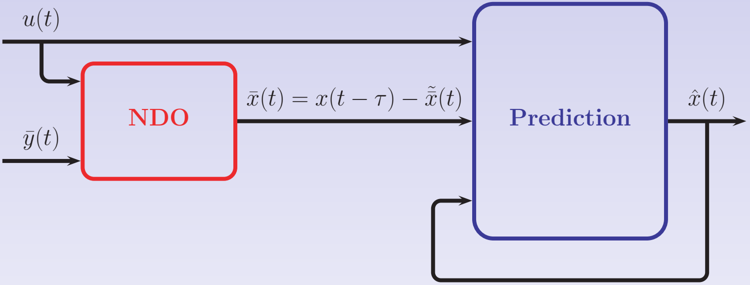
Theorem

Let a ranking of $\mathbf{k}\{W, Z, T\}$ (where T stands for the differential indeterminate corresponding to ζ) be fixed, let it be such that any derivative of the components of W is lower than Z_1, $Z_2, \ldots$, and Z_n whose derivatives are all lower that T. Let $\mathcal{A}$ be a characteristic set of $\mathbf{I}(\mathcal{X})$.

If z is observable with respect to w, then each Z_i is (effectively) introduced in $\mathcal{A}$ by a differential polynomial of order zero and with degree $\leq \mathrm{d}^\circ_w z_i$ in $Z_i \quad (1 \leq i \leq n)$.

Conversely, if each Z_i is introduced in $\mathcal{A}$ by a differential polynomial of order zero and degree d_i in Z_i then z is observable with respect to w, $\mathrm{d}^\circ_w z_i \geq d_i$, and $\mathrm{d}^\circ_w z_i$ divides $d_i \cdots d_2 \cdot d_1$ (hence $\mathrm{d}^\circ_w z_1 = d_1$).

The structure of the new observer may be depicted as in



The numerical differentiation block (NDO) processes the data, here u and y, to yield a delayed estimate of the state. The following notations

$$\check{x} : t \mapsto \check{x}(t) = x(t-\tau)\,,$$
$$\check{u} : t \mapsto \check{u}(t) = u(t-\tau)\,,$$

are adopted so that

$$\check{x} = \bar{x} + \tilde{\bar{x}}\,.$$

The prediction block details are as follows

$$\text{for } t - t_0 \leq \tau\,, \quad \overset{\text{Initialization}}{\begin{cases} \dot{\hat{x}} = f(\hat{x}, u)\,, \\ \hat{x}(t_0) = \hat{x}_0\,, \end{cases}} \tag{5a}$$

$$\text{for } t - t_0 \geq \tau\,, \quad \overset{\text{Model filtering}}{\begin{cases} \dot{\hat{\check{x}}}(t) = f(\bar{x}, \check{u}) + K(\bar{x} - \hat{\check{x}})\,, \\ \hat{\check{x}}(t_0 + \tau) = \hat{x}(t_0 + \tau)\,, \end{cases}} \tag{5b}$$

$$\overset{\text{Prediction}}{\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}\hat{x}(s) = f(\hat{x}(s), u(s))\,, \\ s \in [t - \tau, t]\,, \\ \hat{x}(t - \tau) = \hat{\check{x}}(t)\,. \end{cases}} \tag{5c}$$

The convergence of the observer is global and bounded.

Concluding remarks

- The **differential algebraic approach to observability** is seen as the missing link between **regularized numerical differentiation schemes** and dynamic inversion techniques as implemented in the Kalman filter.
- These two techniques should be combined to provide more general and more practical solutions to the observation problems.