Necessary and sufficient conditions for exponential synchronization of nonlinear systems

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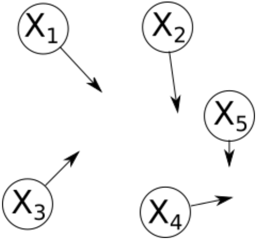
The synchronization problem

Consider N systems or agents



Each system in $\mathbb{R}^n$ satisfies the dynamics $\dot{x}_i = f(x_i) + g(x_i)u_i$

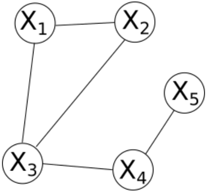
- Same drift vector field f, and same control vector field g
- But not the same controls in $\mathbb{R}^p$
  $\Rightarrow$ So they define different dynamics in the same space ($\mathbb{R}^n$).

The synchronization problem

Synchronization problem : *Find state feedback control law u_i such that each agent converges to each other.*

With control constraints :

1. the control of agent i can use only the knowledge of its state x_i and state of some other agents depending on a **undirected communication graph.**



$$\Rightarrow \begin{array}{rcl} u_1 &=& \phi_1(x_1, x_2, x_3) \\ u_2 &=& \phi_2(x_1, x_2, x_3) \\ u_3 &=& \phi_3(x_1, x_2, x_3, x_4) \\ u_4 &=& \phi_4(x_3, x_4, x_5) \\ u_5 &=& \phi_5(x_4, x_5) \end{array}$$

2. all controls u_i are not used when each agent has found an **agreement**.

$$\phi_i(z, \ldots, z) = 0 \ , \ i = 1, \ldots N, \ \forall z \in \mathbb{R}^n$$

$\Rightarrow$ stabilizing all agents to a steady state is not an appropriate solution.

The synchronization problem

The synchronization manifold $\mathcal{D}$

$$\mathcal{D} = \{(x_1, \ldots, x_N) \in \mathbb{R}^{Nn} \mid x_1 = x_2 \cdots = x_N\}.$$

$\Rightarrow$ Synchronizing means "make $\mathcal{D}$ asymptotically stable".

Let $X(x,t)$ denotes the solutions initiated from $x = (x_1, \ldots, x_N)$ defined for all t in $[0, T(x))$.

Uniform Exponential Stability of a set

- Local : $\exists (r, k, \lambda)$ positive such that $\forall x$ satisfying $|x|_{\mathcal{D}} < r$,

$$|X(x,t)|_{\mathcal{D}} \leq k \exp(-\lambda t)\, |x|_{\mathcal{D}} \ , \ \ t \in [0, T(x))$$

  where $|\cdot|_{\mathcal{D}}$ is the Euclidean distance to the set $\mathcal{D}$.
- Global : $r = +\infty$.

The local synchronization problem

Our synchronization problem formulation

Construct $u_i = \phi_i(x)$, $i = 1 \ldots, N$ such that

1. For all non-communicating pair (i, j) with $i \neq j$

$$\frac{\partial \phi_i}{\partial x_j}(x) = \frac{\partial \phi_j}{\partial x_i}(x) = 0$$

2. $(\phi_i)_{i \in [1,N]}$ is zero on $\mathcal{D}$
3. The manifold $\mathcal{D}$ of the closed-loop system

$$\dot{x}_i = f(x_i) + g(x_i)\phi_i(x), \quad i = 1, \ldots, N$$

   is locally uniformly exponentially stable.

When $r = \infty$, it is called the global exponential synchronization problem.

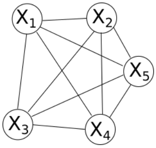
In the following:

1. Some necessary conditions for synchronization
2. Some sufficient conditions for local synchronization
3. Design based on these sufficient conditions
4. Global synchronization
5. Conclusion

Necessary Conditions

First case considered

- All agents can communicate with each other



- The control law preserves some symmetry of the communication topology : Invariance by permutation.

$$\phi_1(x_2, x_1, x_3, x_4, x_5) = \phi_2(x_1, x_2, x_3, x_4, x_5)$$

Theorem (VA-BJ-LP 2016)

Assume some bounds on derivative of f, g, ϕ and that $u = \phi(x)$ solves the local uniform exponential synchronization. Then the following two properties hold

1. Infinitesimal Stabilizability (IS)
2. Control Matrix Function (CMF)

Infinitesimal stabilizability (IS)

The couple (f, g) is such that for the linearized controlled system

$$\dot{\tilde{z}} = \frac{\partial f}{\partial z}(z)\tilde{z} + g(z)\tilde{u} \ , \ \dot{z} = f(z) \ , \ (\tilde{z}, z) \in \mathbb{R}^n \times \mathbb{R}^n \ ,$$

$\{(z, \tilde{z}) \in \mathbb{R}^n \times \mathbb{R}^n, \tilde{z} = 0\}$ is stabilizable by a (linear in $\tilde{z}$) state feedback.

$\Rightarrow$ For linear systems

$$\dot{x}_i = Ax_i + Bu_i \ , i = 1, \ldots N$$

it is equivalent with stabilizability of the couple (A, B).

Control Matrix Function (CMF)

The couple (f, g) is such that for all $Q > 0$, there exist $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $\underline{p}\,\mathrm{Id}_n \leq P(z) \leq \overline{p}\,\mathrm{Id}_n$ and

$$\text{if } \tilde{z}^\top P(z) g(z) = 0 \Rightarrow \underbrace{\frac{\partial \tilde{z}^\top P(z) \tilde{z}}{\partial z} f(z) + 2\tilde{z}^\top P(z) \frac{\partial f}{\partial z}(z)\tilde{z}}_{= \overbrace{\tilde{z}^\top P(z) \tilde{z}}^{\cdot} \text{ along the linearized syst.}} \leq -\tilde{z}^\top Q \tilde{z}$$

Picking $V(z, \tilde{z}) = \tilde{z}^\top P(z)\tilde{z}$, this property is a CLF property which characterizes the Infinitesimal stabilizability property for

$$\dot{\tilde{z}} = \frac{\partial f}{\partial z}(z)\tilde{z} + g(z)\tilde{u} \ , \ \dot{z} = f(z)$$

Sketch of the proof :

Let $z = x_1$, $e_i = x_i - x_1$, $i = 2, \dots N$, the system can be rewritten

$$\dot{e} = F(e, z) \;\; \dot{z} = G(e, z)$$

Exponential Synchronization

$$\Updownarrow$$

$\mathcal{D} := \{(z, e) : \; e = 0\}$ exponentially stable

How can we characterize transverse exponential stability ?

Sketch of the proof :

Back to the basis : Consider the (sufficiently smooth) system:

$$\dot{e} = F(e)\ ,\ \ F(0) = 0\ ,\ \ e \in \mathbb{R}^{n_e}$$

Basic Lyapunov Theorem

The following three properties are equivalent:

- Local exponential stability
- Global exponential stability of the linearized system

$$\dot{\tilde{e}} = \frac{\partial F}{\partial e}(0)\tilde{e}$$

- Algebraic Lyapunov matrix inequality

$$\forall Q > 0\ ,\ \exists \mathcal{P} > 0\ ,\ \mathcal{P}\frac{\partial F}{\partial e}(0) + \frac{\partial F}{\partial e}(0)^\top \mathcal{P} < -Q$$

Consider a (sufficiently smooth) (z, e)-system in the form:

$$\dot{e} = F(e, z) \ , \ \dot{z} = G(e, z) \ , \ e \in \mathbb{R}^{n_e} \ , \ z \in \mathbb{R}^{n_z}$$

such that the manifold $\mathcal{D} := \{(z, e) : \ e = 0\}$ is invariant $F(0, z) = 0$, $\forall z$.

Transverse Lyapunov Theorem (VA-BJ-LP-2013)

Assuming some bounds on derivatives of F and G, the following three properties are equivalent:

- Local uniform exponential stability of $\mathcal{D} := \{(z, e) : \ e = 0\}$
- Global exponential stability of $\tilde{\mathcal{D}} := \{(z, \tilde{e}) : \ \tilde{e} = 0\}$ along the transversally linearized system

$$\dot{\widetilde{e}} = \frac{\partial F}{\partial e}(0, z)\widetilde{e} \ , \ \dot{z} = G(0, z)$$

- Algebraic Lyapunov matrix inequality for all $\mathcal{Q}$ there exists $\underline{p}\, I \leq \mathcal{P}(z) \leq \overline{p}\, I$ such that

$$\frac{\partial \widetilde{e}^\top \mathcal{P}(z)\widetilde{e}}{\partial z} G(0, z) + 2\widetilde{e}^\top \mathcal{P}(z) \frac{\partial F}{\partial e}(0, z)\widetilde{e} \leq -\widetilde{e}^\top \mathcal{Q}\widetilde{e}$$

Sketch of the proof

- The synchronized system in closed loop is

$$\begin{aligned}\dot{e}_i = F_i(e,z) = f(e_i+z) - f(z) \\ + g(e_i+z)\phi_i(z, e_1+z, \dots, e_N+z) \\ - g(z)\phi_1(z, e_1+z, \dots, e_N+z)\end{aligned}$$

- With the Trans. Lyap. Theo., $\tilde{\mathcal{D}} := \{(z,\tilde{e}) : \tilde{e}=0\}$ is exponentially stable along

$$\dot{\widetilde{e}} = \frac{\partial F}{\partial e}(0,z)\widetilde{e}\ ,\ \dot{z} = G(0,z)$$

- With the invariance by permutation assumption:

$$\dot{\widetilde{e}}_i = \frac{\partial F}{\partial e}(0,z)\widetilde{e} = \frac{\partial f}{\partial x}(z)\widetilde{e}_i + g(z)\left[\frac{\partial \phi_i}{\partial x_i}(z,\dots,z) - \frac{\partial \phi_1}{\partial x_i}(z,\dots,z)\right]\widetilde{e}_i$$

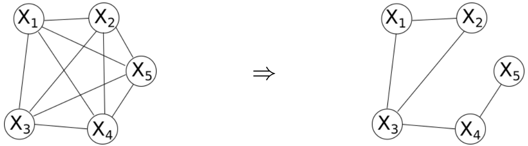
$\Rightarrow$ Infinitesimal stabilizability and CMF hold

Necessary conditions

If we assume invariance by permutation, two necessary conditions

1. Infinitesimal Stabilizability (IS)
2. Control Matrix Function (CMF)

Is it still true when removing symmetry property and considering communication constraints ?



Necessary conditions

Theorem (VA-BJ-ST-2017)

Assume some bounds on derivative of f, g, ϕ and that $u = \phi(x)$ solves the local uniform exponential synchronization over a graph $\mathcal{G}$. Then properties **IS** and **CMF** hold.

Note that no invariance by permutation is now assumed.

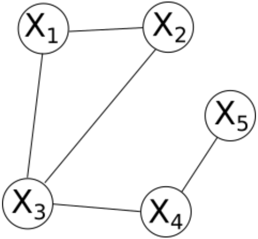
In the following:

1. Some necessary conditions for synchronization
2. Some sufficient conditions for local synchronization
3. Design based on these sufficient conditions
4. Global synchronization
5. Conclusion

Sufficient conditions

Given a graph $\mathcal{G}$ we introduce the **Laplacian matrix** in $\mathbb{R}^{N \times N}$:

$$L_{ii} = \sum_{j \neq i} L_{ij} \ , \ L_{ij} = -1 \text{ if } (i,j) \in \mathcal{E}$$



$$\Rightarrow L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Sufficient condition

For linear system

$$\dot{x}_i = Ax_i + Bu_i \ , \ i = 1, \dots N$$

Stabilizability + Connectivity $\Rightarrow$ Synchronization.

Theorem : Linear case (Scardovi-Sepulchre-2009)

If

- (A, B) is stabilizable. i.e. there exists K such that $A + BK$ is stable.
- the graph $\mathcal{G}$ is connected with associated Laplacian matrix $L = (L_{ij})$.

Then the control law

$$\phi_i(x) = \sum_{j=1}^{N} L_{ij} K x_j$$

solves the (global) exponential synchronization problem.

Sufficient condition

Control Matrix Function Integrability + Connectivity $\Rightarrow$ (local) Synchronization.

Theorem (VA-BJ-ST-2015)

Assume that $\exists$ a C^2 function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$

- The **CMF'** condition holds.
- $\exists U : \mathbb{R}^n \to \mathbb{R}$, $\alpha : \mathbb{R}^n \to \mathbb{R}^p$ such the **integrability** holds, i.e.

$$\frac{\partial U}{\partial z}(z)^\top = P(z)g(z)\alpha(z)$$

Then, given a **connected** graph $\mathcal{G}$ there exists ℓ such that

$$\phi_i(x) = -\ell\alpha(x_i)\sum_{j=1}^{N} L_{ij}U(x_j)\ ,\ x \in \mathbb{R}^{nm}$$

solves the local uniform exponential synchronization problem.

About the CMF' condition

The CMF necessary condition was

$$\text{if } v^\top P(z)g(z) = 0 \Rightarrow \frac{\partial v^\top P(z)v}{\partial x} + 2v^\top P(z)\frac{\partial f}{\partial z}(z)v \leq -v^\top Qv$$

and the CMF' sufficient condition is (forgetting the function α)

$$\frac{\partial v^\top P(z)v}{\partial x} + 2v^\top P(z)\frac{\partial f}{\partial z}(z)v - \rho v^\top P(z)g(z)g(z)^\top P(z)v \leq -v^\top Qv \ ,$$

In the **linear context** the first one becomes:

$$\text{if } v^\top PB = 0 \Rightarrow 2v^\top A^\top Pv \leq -v^\top Qv$$

and the second one

$$PA + A^\top P - \rho PBB^\top P \leq -Q \ ,$$

- With Finsler's Lemma we get equivalence in the linear context.
- If z is in a compact set, equivalence is also obtained in the nonlinear context.

About the synchronizing control law

The control law is :

$$\phi_i(x) = -\ell\alpha(x_i) \sum_{j=1}^{N} L_{ij} U(x_j)$$

For all $x = (z, \ldots, z)$ in $\mathcal{D}$

$$\frac{\partial \phi}{\partial x}(x) = -\ell\alpha(z) \frac{\partial U}{\partial z}(z) \otimes L \ .$$

$\Rightarrow$ This is a direct extension of Scardovi-Sepulchre-2009.

Sketch of the proof

- The closed loop system is :

$$x_i = f(x_i) - \ell g(x_i)\alpha(x_i) \sum_{j=1}^{N} L_{ij} U(x_j) \ , \ i = 1, \dots, N$$

- Let $z = x_1$, $e_i = x_i - x_1$, the system can be rewritten

$$\dot{e} = F(e, z) \ \ \dot{z} = G(e, z)$$

- We want ℓ such that $\tilde{\mathcal{D}} := \{(z, \tilde{e}) : \ \tilde{e} = 0\}$ is exponentially stable along the transversally linearized system

$$\dot{\widetilde{e}} = \frac{\partial F}{\partial \widetilde{e}}(0, z)\widetilde{e} \ , \ \ \dot{z} = G(0, z)$$

(Because of the Transverse exponential stability theorem VA-BJ-LP-2013)

Sketch of the proof

- With the Integrability Condition

$$\frac{\partial F}{\partial e}(0,z) = \mathrm{Id}_{N-1} \otimes \frac{\partial f}{\partial z}(z) + \ell A \otimes \left(g(z)\alpha(z) \quad \underbrace{\frac{\partial U}{\partial x}(z)}_{(P(z)g(z)\alpha(z))^\top} \right)$$

where A is a matrix which depends on L.

- If the graph is Connected, A is Hurwitz,

$$\exists S > 0 \ , \ \nu > 0 \ , \ SA + A^\top S \leq -\nu S$$

- With CMF', it yields with $\mathcal{P}(z) = S \otimes P(z)$ and ℓ sufficiently large

$$\frac{\partial \tilde{e}^\top \mathcal{P}(z)\tilde{e}}{\partial z} G(0,z) + 2\tilde{e}^\top \mathcal{P}(z)\frac{\partial F}{\partial e}(0,z)\tilde{e} \leq -\tilde{e}^\top Q \tilde{e}$$

$\Rightarrow e^\top \mathcal{P}(z) e$ is a Lyapunov function $\Rightarrow$ Local Synchronization

Sufficient conditions

Conclusion : Given vector fields f and g, to construct a synchronizing control law, one needs to find a matrix function P

- which satisfies the CMF condition.
- which satisfies the integrability condition.

Question : Is is possible ?

In the following:

1. Some necessary conditions for synchronization
2. Some sufficient conditions for local synchronization
3. Design based on these sufficient conditions
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Design based on CMF and Integrability

Assume we have found a matrix function $P_a(z_a)$ which satisfies the CMF and integrability conditions for the z_a subsystem,

$$\dot{z}_a = f_a(z_a) + g_a(z_a)u_a \ , \ z_a \in \mathbb{R}^{n_a}$$

Question : is it possible to construct P that satisfies the CMF and integrability conditions for the following $\mathbb{R}^{n_a+1}$ system ?

$$\dot{z}_a = f_a(z_a) + g_a(z_a)z_b, \ \dot{z}_b = f_b(z) + g_b(z)u \ , \ z = (z_a, z_b) \in \mathbb{R}^{n_a+1}$$

with $0 < \underline{g}_b \leq g_b(z) \leq \overline{g}_b$

$\Rightarrow$ This is the context of the backstepping

Design based on CMF and Integrability

Killing controlled vector field assumption (KCV)

There exists a non-vanishing smooth function $q_a : \mathbb{R}^{n_a} \to \mathbb{R}$ such that

$$\frac{\mathfrak{d}_{g_a} P_a(z_a)}{q_a(z_a)} + 2\frac{P_a(z_a)}{q_a(z_a)}\frac{\partial g_a}{\partial z_a}(z_a) - 2P_a(z_a)\frac{g_a(z_a)}{q_a(z_a)^2}\frac{\partial q_a}{\partial z_a}(z_a) = 0$$

With $\tilde{g}_a(z_a) = \frac{g(z_a)}{q_a(z_a)}$, this assumption means

$$\overbrace{\tilde{z}_a^\top P(z_a)\tilde{z}_a}^{\cdot} = 0$$

along the linearized system

$$\dot{\tilde{z}}_a = \frac{\partial \tilde{g}_a}{\partial z_a}(z_a)\tilde{z}_a \ , \ \dot{z}_a = \tilde{g}_a(z_a)$$

Design based on CMF and Integrability

Theorem (VA-BJ-ST-2016)

Assume that there exists P_a that satisfies the CMF, the Integrability and the KCV conditions for z_a in $\mathcal{C}_a$.
Then for all positive real number M_b, the overall system satisfies the CMF, the Integrability and the KCV conditions in the compact set $\mathcal{C}_a \times [-M_b, M_b] \subset \mathbb{R}^{n_a+1}$ with the symmetric covariant tensor field P_b be given by

$$P_b(z) = \begin{bmatrix} P_a(z_a) + S_a(z)S_a(z)^\top & S_a(z)q_a(z_a) \\ S_a(z)^\top q_a(z_a) & q_a(z_a)^2 \end{bmatrix}$$

where

$$S_a(z) = \frac{\partial q_a}{\partial z_a}(z_a)^\top z_b + \eta\alpha_a(z_a)P_a(z_a)g_a(z_a)$$

and η is a positive real number.

We propagate the property !

In the following:

1. Some necessary conditions for synchronization
2. Some sufficient conditions for local synchronization
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Global synchronization

First, consider two agents

$$\dot{x}_1 = f(x_1) + g(x_1)u_1 \ , \ \dot{x}_2 = f(x_2) + g(x_2)u_2$$

Assume that $\exists$ a C^2 function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$

- The **CMF** condition holds.
- $\exists U : \mathbb{R}^n \to \mathbb{R}$, $\alpha : \mathbb{R}^n \to \mathbb{R}^p$ such the **integrability Condition** holds, i.e.

$$\frac{\partial U}{\partial z}(z)^\top = P(z)g(z)\alpha(z)$$

Can we get global synchronization ?

Yes, but $e^\top P(z)e$ is not a good candidate Lyapunov function !

P may be used to define a Riemanian metric on $\mathbb{R}^n$.

A Lyapunov function for the system may be constructed:

- we can define the length of any piece-wise C^1 path $\gamma : [s_1, s_2] \to \mathbb{R}^n$ between x_1 and x_2 in as :

$$L(\gamma)\Big|_{s_1}^{s_2} = \int_{s_1}^{s_2} \sqrt{\frac{d\gamma}{ds}(\sigma)^\top P(\gamma(\sigma))\frac{d\gamma}{ds}(\sigma)}\, d\sigma$$

- By minimizing along all such path we get a function $V(x_1, x_2)$.
- Also

$$0 < \sqrt{\underline{p}}|x_1 - x_2| \leq V(x_1, x_2) \leq \sqrt{\overline{p}}|x_1 - x_2|$$

To get a result, we need one more assumption.

Totally geodesics level set of U

For all $(\tilde{z}, z)$ in $S \times \mathbb{R}^n$ such that

$$\frac{\partial U}{\partial x}(x)\tilde{z} = 0, \tilde{z}^\top P(z)\tilde{z} = 1 \ ,$$

any geodesic γ, with $\gamma(0) = z$ and $\frac{d\gamma}{ds}(0) = \tilde{z}$ satisfies

$$\frac{\partial U}{\partial z}(\gamma(s))\frac{d\gamma}{ds}(s) = 0 \ , \ \forall s$$

In the case of two agents:

Theorem (VA-BJ-LP-2016)

Assume that $\exists$ a C^4 function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$

- The **CMF** condition holds.
- $\exists U : \mathbb{R}^n \to \mathbb{R}$, $\alpha : \mathbb{R}^n \to \mathbb{R}^p$ such the **integrability** holds, i.e.

$$\frac{\partial U}{\partial z}(z)^\top = P(z)g(z)\alpha(z)$$

- The level set of U are **totally geodesic** (with respect to $(\mathbb{R}^n, P)$)

Then, there exists a function ℓ such that

$$\phi_1(x) = -\ell(x_1, x_2)\alpha(x_1)\left[U(x_2) - U(x_1)\right]$$
$$\phi_2(x) = -\ell(x_1, x_2)\alpha(x_2)\left[U(x_1) - U(x_2)\right]$$

solves the global uniform exponential synchronization problem.

Sketch of the proof

- Consider two agents with g constant

$$\dot{x}_1 = f(x_1) + gu_1 \ , \ \dot{x}_2 = f(x_2) + gu_2$$

- Assume that there exists P (constant) such that $\underline{p}\,\mathrm{Id}_n \leq P \leq \overline{p}\,\mathrm{Id}_n$ and

$$\text{if } \tilde{z}^\top P g = 0 \Rightarrow \tilde{z}^\top P \frac{\partial f}{\partial z}(z)\tilde{z} \leq -\tilde{z}^\top Q \tilde{z} \qquad = CMF$$

- In this case the Integrability Condition is trivially satisfied

$$U(z) = g^\top P z$$

- Let $e = x_2 - x_1$, $z = x_1$, note that we have

$$\begin{aligned} e^\top P \dot{e} &= e^\top P \left[f(z+e) - f(z) - \ell g g^\top P e \right] \\ &= \int_0^1 e^\top \left[P \frac{\partial f}{\partial e}(z+se) - \ell P g g^\top P \right] e \ ds \end{aligned}$$

Case with N agent in the Euclidean case

Theorem (VA-BJ-LP-2016)

1. Assume that $g(z) = G$ and that $\exists$ a matrix P in $\mathbb{R}^{n\times n}$ such that **CMF** condition holds.
2. the graph is connected with Laplacian matrix L.

Then there exist ℓ and positive real numbers $c_1, \dots, c_N$ such that

$$\phi_i(x) = -\ell\, c_i \sum_{j=1}^{N} L_{ij} G^\top P x_j$$

solves the global uniform exponential synchronization problem.

Open question : What about the Riemannian case with N agent ?

Conclusion

In conclusion:

- We have formulized a synchronization problem.
- Assume some constraints on the control, infinitesimal stabilizability and its Lyapunov characterization (CMF) are necessary condition.
- Adding an integrability property, these condition becomes sufficient.
- An iterative construction inspired from the backstepping can be introduced.
- Global result is possible if
  - there are only 2 agents + totally geodesic assumption.
  - Constant control vector field + Euclidean stabilizability property

On going work:

- Construction of synchronizing controller for monotonic nonlinear systems.

Open questions:

- Global result for more then 2 agents in the Riemannian case.
- what about directed graphs ?

Reference on the topic

Presentation based on these publications:

- V. Andrieu, B. Jayawardhana, L. Praly, *On the transverse exponential stability and its use in incremental stability, observer and synchronization*, IEEE CDC 2013
- V. Andrieu, B. Jayawardhana, L. Praly, *Transverse exponential stability and applications*, To appear in IEEE TAC 2016
- V. Andrieu, B. Jayawardhana, S. Tarbouriech, *Necessary and sufficient conditions for local exponential synchronization of nonlinear systems*, IEEE CDC 2015
- V. Andrieu, B. Jayawardhana, S. Tarbouriech, *Exponential synchronization of nonlinear systems*, To appear in IEEE TAC

As an illustrative example, consider the case in which the vector fields f and g are given by

$$f(z) = \begin{bmatrix} -z_{a1} + \sin(z_{a2})\cos(z_{a1}) + z_{a2} \\ [2 + \sin(z_{a1})]z_b \\ 0 \end{bmatrix} ,$$

$$g(z) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

This system may be rewritten with $z_a = (z_{a1}, z_{a2})$ as

$$\dot{z}_a = f_a(z_a) + g_a(z_a)z_b \ , \ \dot{z}_b = u$$

with

$$f_a(z_a) = \begin{bmatrix} -z_{a1} + \sin(z_{a2})\cos(z_{a1}) + z_{a2} \\ 0 \end{bmatrix} ,$$

$$g_a(z_a) = \begin{bmatrix} 0 \\ 2 + \sin(z_{a1}) \end{bmatrix}$$

Consider the matrix P_a given as

$$P_a = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

- With

$$U(z_a) = z_{a1} + 2z_{a2} \ ,$$

The integrability condition is satisfied with $\alpha_a = \frac{1}{2+\sin(z_{a1})}$.

- We have

$$v^\top \frac{\partial U}{\partial z_a}(z_a) = 0 \Leftrightarrow v_1 + 2v_2 = 0$$

Moreover, we have

$$\begin{bmatrix} -2 & 1 \end{bmatrix} P_a \frac{\partial f_a}{\partial z_a}(z_a) \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -3 \left[-2 \frac{\partial f_{a1}}{\partial z_{a1}} + \frac{\partial f_{a1}}{\partial z_{a2}} \right]$$

$$= -3. \\ [-2(-1 + \sin(z_{a2}) \sin(z_{a1})) - \cos(z_{a1}) \cos(z_{a2}) + 1]$$

$$\leq -3$$

Hence the CMF condition is satisfied.

Design based on CMF and Integrability

- Finally the KCV condition is satisfied by taking $q_a(z_a) = 2 + \sin(z_{a1})$.

$\Rightarrow$ There exist positive real numbers ρ_b and η such that with

$$U(z) = \eta(z_{a1} + 2z_{a2}) + \frac{z_b}{2 + \sin(z_{a1})}$$

and with $\alpha(z) = 2 + \sin(z_{a1})$, the control law

$$\phi_i(x) = -\ell\alpha(x_i) \sum_{j=1}^{N} L_{ij} U(x_j)$$

solve the local exponential synchronization problem for the N identical systems that exchange information via any undirected communication graph $\mathcal{G}$, which is connected.