Qualitative parameter estimation for a class of relaxation oscillators

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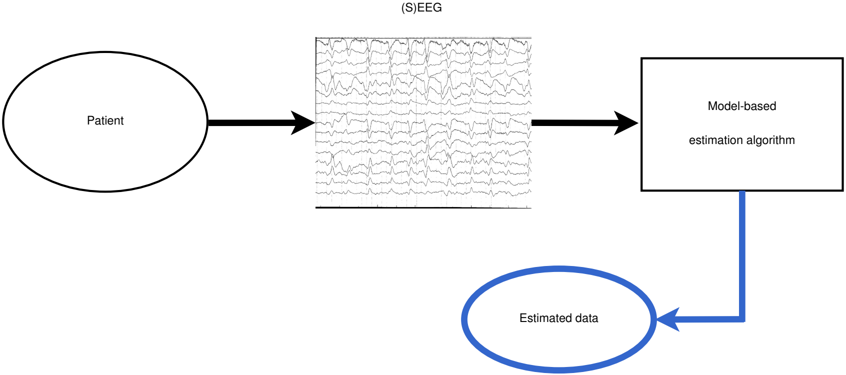
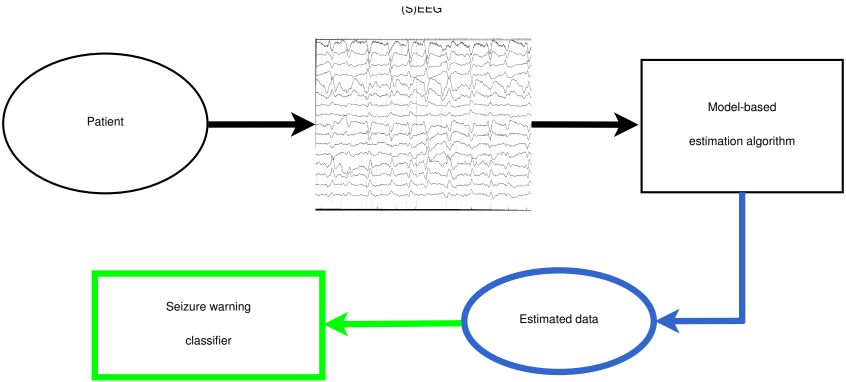
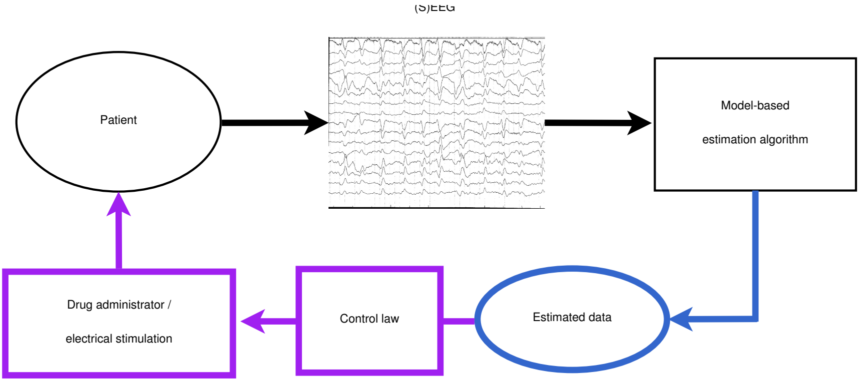
Motivations




(S)EEG
Patient
Model-based
estimation algorithm
Estimated data
Seizure warning
classifier


(S)EEG
Patient
Model-based
estimation algorithm
Estimated data
Control law
Drug administrator /
electrical stimulation

Peculiarities of neuronal dynamics

- Redundancy and large variability of biophysical parameters

  $\Rightarrow$Disparate combinations of parameters lead to the same activity pattern
- Sharp transitions between different qualitative activity modes due to slowly varying parameters

Existing nonlinear estimation methods

- Stochastic methods

  ◦ Schiff et al. 2008, $\rightarrow$ Unscented Nonlinear Kalman filter
  Applied to a spatiotemporal model of the cortex
  Lack of convergence proof

- Deterministic schemes

  ◦ Chong et al. 2015, $\rightarrow$ Supervisory observer
  Convergence guanrantees
  Applied to a neural mass model
  Choice on the unknown parameters relies on prior simulation analysis

Our objectives

- Online parameter estimation
  $\rightarrow$ peculiarities of neuronal dynamics
- Convergence analytic guarantees

Outline

Outline

A class of two-time-scale nonlinear systems

Consider the following nonlinear system

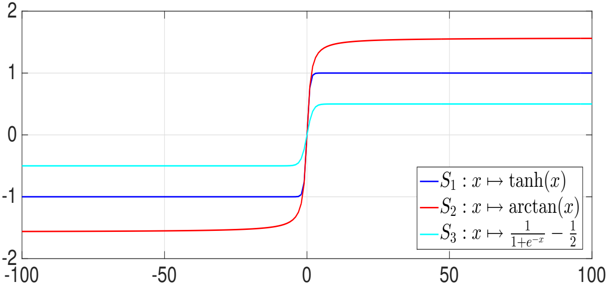
$$\begin{cases} \dot{x}_f = -x_f + S(\beta x_f + u - x_s), & \text{(1a)} \\ \dot{x}_s = \varepsilon(x_f - x_s), & \text{(1b)} \end{cases}$$

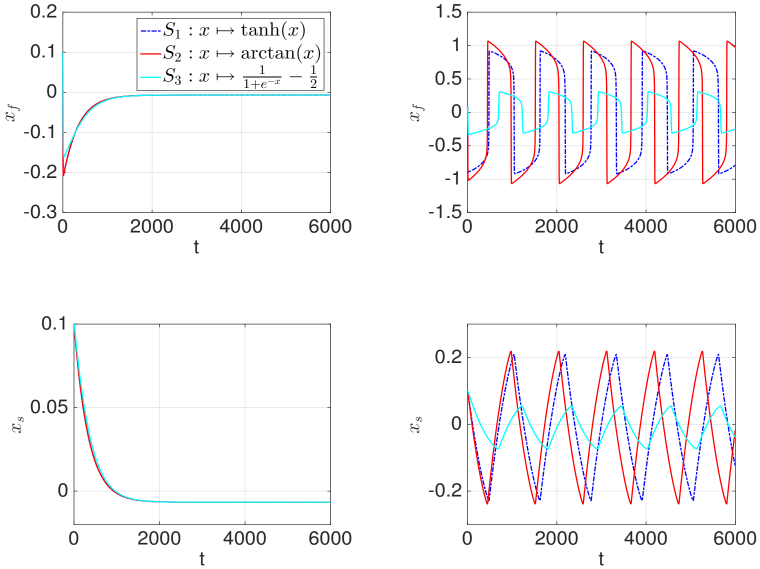
where

- $x_f, x_s \in \mathbb{R}$ are known state variables,
- $\beta \in \mathbb{R}$ is an unknown parameter,
- $u \in \mathbb{R}$ is a known *constant* input,
- $0 < \varepsilon \ll 1$ is an unknown parameter,
- $S : \mathbb{R} \to \mathbb{R}$ is an unknown sigmoid function.

Assumption on the sigmoid function S

a) S is smooth.
b) $S(0) = 0$.
c) $S'(a) > 0$ for all $a \in \mathbb{R}$ (monotonicity),
$\operatorname{argmax}_{a\in\mathbb{R}} S'(a) = 0$ (sector-valued).
d) $\operatorname{sgn}(S''(a)) = -\operatorname{sgn}(a)$ for all $a \in \mathbb{R}$. ∘





(a) $\beta < \beta_c$ (b) $\beta > \beta_c$

Setting $\epsilon = 0$ in equation (1b), we get the layer dynamics

$$\begin{cases} \dot{x}_f = -x_f + S(\beta x_f + u - x_s), & (2a) \\ \dot{x}_s = 0. & (2b) \end{cases}$$

Using time scale $\tau = \epsilon t$, system (1) becomes

$$\begin{cases} \varepsilon x_f' = -x_f + S(\beta x_f + u - x_s), & (3a) \\ x_s' = x_f - x_s, & (3b) \end{cases}$$

where $'$ stands for $\frac{d}{d\tau}$. Setting $\varepsilon = 0$ in equation (3a), we obtain the reduced dynamics

$$\begin{cases} 0 = -x_f + S(\beta x_f + u - x_s), & (4a) \\ x_s' = x_f - x_s. & (4b) \end{cases}$$

The reduced dynamics defines a one-dimensional vector field over the *critical manifold*

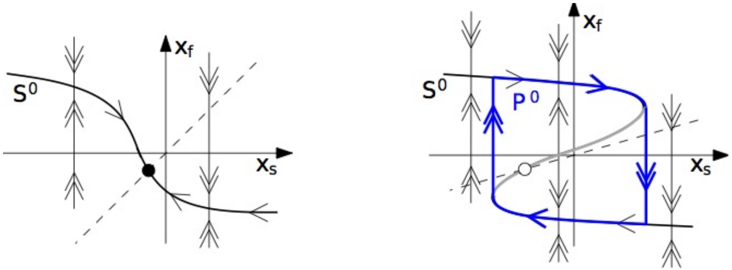
$$S^0 := \{(x_s, x_f) \in \mathbb{R}^2 : \; -x_f + S(\beta x_f + u - x_s) = 0\}.$$

$$\left.\begin{array}{r}\text{Layer dynamics (far from } S^0\text{)}\\ \text{Reduced dynamics (close to } S^0\text{)}\end{array}\right. \Rightarrow \text{Full dynamics}$$

Let

$$\beta_c := \frac{1}{S'(0)},$$

which is strictly positive.



(a) $\beta < \beta_c$ (b) $0 < \beta - \beta_c < 1$

Figure 3: Singular phase portrait.

Stability analysis of system (1)

Proposition 1

1) $\forall \beta < \beta_c$ *and constant input u,* $\exists\ \bar{\varepsilon} > 0$ *s.t. for all* $\varepsilon \in (0, \bar{\varepsilon}]$*:*
◦ *system (1) has a GES fixed point.*
◦ *all trajectories converge in an* $\mathcal{O}(\varepsilon)$*-time to an* $\mathcal{O}(\varepsilon)$*-neighborhood of* S^0*.*

2) $\forall 0 < \beta - \beta_c < 1$, $\exists\ \bar{u} > 0$, *s.t.* $\forall$ *constant input* $u \in (-\bar{u}, \bar{u})$, $\exists\ \bar{\varepsilon} > 0$ *s.t.* $\forall \varepsilon \in (0, \bar{\varepsilon}]$*:*
◦ *system (1) has an almost GAS and LES periodic orbit* P^ε*.*
◦ *almost all trajectories converge to an* $\mathcal{O}(\varepsilon)$*-neighborhood of* S^0 *for almost all time.*

GES = Globally Exponentially Stable, GAS = Globally Asymptotically Stable,

LES = Locally Exponentially Stable

Standard singular perturbation approach in control
[Kokotović et al., 1986], [Khalil, 2002]

- Applicable to the case $\beta < \beta_c$
- Not applicable to the case $\beta > \beta_c$
  - Due to the singularities, equation (4a) no longer has isolated roots.

Proof of Proposition 1 relies on differential geometry arguments
[Fenichel 1979a], [Krupa 2001a, 2001b]

Properties of system (1):

- Two different activities:
  fixed point (resting) or periodic orbit (oscillation)
- Single parameter $\rightarrow$ Transition of the activities

We aim to:

- ◦ Online detect the actual type of activity of system (1)

How to achieve:

× Estimate of the asymptotic value of $\beta - \beta_c$
(ill-posed: unknown S and ϵ)

✓ *Qualitative* estimate $\beta - \beta_c$ (without knowing β_c)

$$\begin{cases} \beta - \beta_c < 0 \rightarrow \text{negative estimate} \\ \beta - \beta_c > 0 \rightarrow \text{positive estimate} \\ \beta \text{ near } \beta_c \rightarrow \text{estimate near zero} \end{cases}$$

Outline

Estimator design

Thanks to singularity theory [1,2], the shape of the critical manifold S^0 is the same as that of the set

$$\{(x_f, x_s) : -x_f^3 + (\beta - \beta_c)x_f + u - x_s = 0\}.$$

⋆ Independent of S and ε
⋆ Fully determined by the sign of $\beta - \beta_c$

We propose the following nonlinear parameter estimator

$$\dot{\hat{\beta}} = -kx_f(-x_f^3 + \hat{\beta}x_f + u - x_s), \qquad (5)$$

where $k > 0$ is a design parameter.

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[1] [Golubitsky and Schaeffer, 1985]
[2] [Franci and Sepulchre, 2014]

Steady-state properties of the qualitative estimator

We implicitly define the function $\hat{\beta}^*(x_f, x_s, u)$ such that

$$-x_f^3 + \hat{\beta}^*(x_f, x_s, u)x_f + u - x_s = 0, \tag{6}$$

for $u, \beta \in \mathbb{R}$ and x_f, x_s on S^0,

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$$x_s = -S^{-1}(x_f) + \beta x_f + u. \tag{8}$$

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Replacing this expression for x_s in (6), we obtain

$$\hat{\beta}^*(x_f, x_s, u) = \frac{x_f^3 - S^{-1}(x_f) + \beta x_f}{x_f}.$$

We write

$$\hat{\beta}^*(x_f, \beta) \quad = \quad \frac{x_f^3 - S^{-1}(x_f) + \beta x_f}{x_f}.$$

Lemma 1

1) $\hat{\beta}^*$ *is smooth on* $S(\mathbb{R}) \times \mathbb{R}$.

2) When $\beta < \beta_c$, $\frac{\partial \hat{\beta}^*(x_f,\beta)}{\partial \beta} > 0$ *for* $x_f \in S(\mathbb{R})$.

3) When $0 < \beta - \beta_c < 1$, $\frac{\partial \hat{\beta}^*(x_f,\beta)}{\partial \beta} > 0$*:*

- ◦ *for* x_f *in a neighborhood of the origin*
- ◦ *for large* x_f*, if a sufficient condition is satisfied.*

4) $\hat{\beta}^*(x_f, \beta) = \beta - \beta_c + \mathcal{O}(x_f^2)$ *for any* $x_f \in S(\mathbb{R})$ *and* $\beta \in \mathbb{R}$. □

Stability analysis of the estimator

Proposition 2

$\forall \Delta > 0$, $\exists$ *functions* $\ell_\Delta \in \mathcal{KL}$ *and* $\gamma_\Delta \in \mathcal{K}_\infty$ *s.t if*
i) x_{fi} *is PE,*
ii) $\max(\|x_{fi}\|_\infty, \|x_{si}\|_\infty, \|u_i\|_\infty) \leqslant \Delta$, *where* $i \in \{1,2\}$.
Then for any $|\hat{\beta}_i(0)| \leqslant \Delta$, *the corresponding solution* $\hat{\beta}_i$ *to system (5) satisfies for all* $t \geqslant 0$

$$\begin{aligned} |\hat{\beta}_1(t) - \hat{\beta}_2(t)| \leqslant \; & \ell_\Delta\left(|\hat{\beta}_1(0) - \hat{\beta}_2(0)|, t\right) + \gamma_\Delta\big(\|x_{f1} - x_{f2}\|_{[0,t)} \\ & + \|x_{s1} - x_{s2}\|_{[0,t)} + \|u_1 - u_2\|_{[0,t)}\big). \end{aligned}$$

$\square$

PE=Persistency of Excitation

Main idea of the proof of Proposition 2

We rewrite the estimator (5) as

$$\underbrace{\dot{\hat{\beta}} = -kx_f^2\hat{\beta} + \underbrace{\left(kx_f^4 + kx_f(x_s - u)\right)}_{input\ U(t)}}_{LTV\ system}.$$

$$\left.\begin{array}{l} x_f - PE \\ \text{boundedness of } x_f, x_s, u \end{array}\right\} \overset{\text{Lyapunov analysis}}{\Longrightarrow} \text{desired result}$$

PE condition for system (1)

Lemma 2

For both $\beta < \beta_c$ and $0 < \beta - \beta_c < 1$, the corresponding solution to system (1) satisfies

i) x_f is PE (in the semiglobal sense). □

We consider the overall system which is three-dimensional and given by

$$
\begin{cases}
\dot{x}_f = -x_f + S(\beta x_f + u - x_s), & \text{(9a)} \\
\dot{x}_s = \varepsilon (x_f - x_s), & \text{(9b)} \\
\dot{\hat{\beta}} = -k x_f (-x_f^3 + \hat{\beta} x_f + u - x_s). & \text{(9c)}
\end{cases}
$$

We denote the corresponding solution to (9)

$$
z(t) = \left(x_f(t), x_s(t), \hat{\beta}(t) \right).
$$

Stability analysis of the overall system

Theorem 1

1) $\forall \beta < \beta_c, \Delta > \delta > 0$*, and constant input u satisfying* $\delta < |u| < \Delta$*,* $\exists \bar{\varepsilon} > 0$ *s.t* $\forall \varepsilon \in (0, \bar{\varepsilon}]$*, system (9) has a SGES fixed point* $\bar{z}$*.*

2) $\forall 0 < \beta - \beta_c < 1, \Delta > 0$*,* $\exists \bar{u} \in (0, \Delta)$ *s.t.* $\forall u \in (-\bar{u}, \bar{u})$*,* $\exists \bar{\varepsilon} > 0$*, s.t.* $\forall \varepsilon \in (0, \bar{\varepsilon}]$*, system (9) has an almost GAS and LES periodic orbit* Q^{ε}*.*

SGES = Semi Globally Exponentially Stable, GAS = Globally Asymptotically Stable,

LES = Locally Exponentially Stable

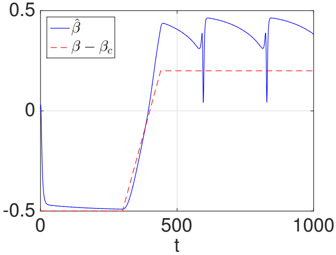
Theorem 2

Consider system (9).

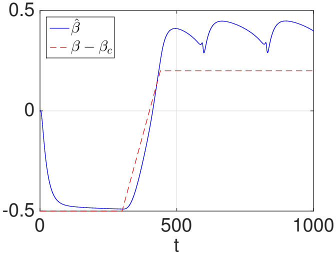
1. *For any* $\beta < \beta_c$*, let* x_f^* *be the* x_f*-component of the fixed point of (9). Then* $\hat{\beta}(t)$ *converges to* $\hat{\beta}^*(x_f^*, \beta)$ *as time goes to infinity.*
2. *For any* $0 < \beta - \beta_c < 1$*, let* x_f^{lc} *be the* x_f*-component of the periodic orbit of (9), which is* T^ε*-periodic.*
   Then $\hat{\beta}(t)$ *converges to an* $O(\varepsilon)$*-neighborhood of* $\hat{\beta}^*(x_f^{lc}(t), \beta)$ *for most time.*

Outline

Evolution of $\hat{\beta}$ with different values of β

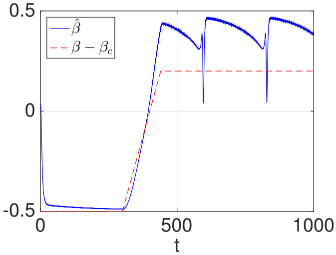


(a) simulation without filter



(b) simulation with filter

With small output measurement noises



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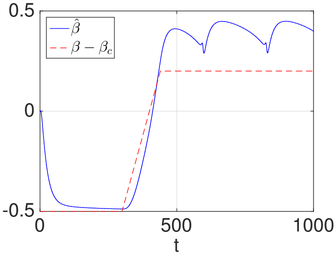
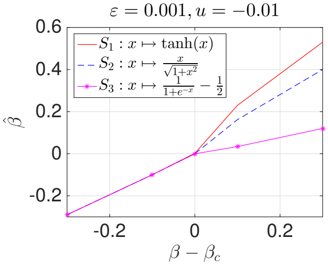
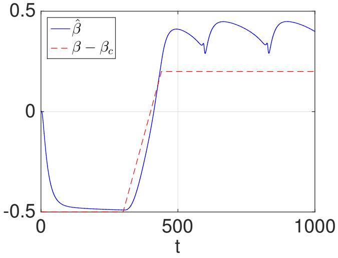


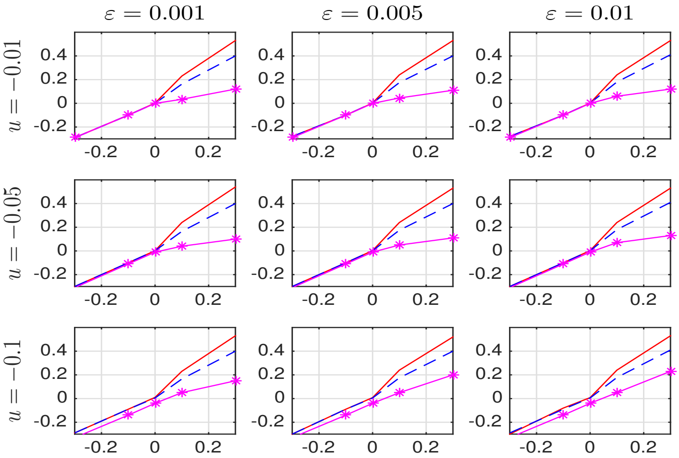
(b) simulation with filter

Evolution of $\hat{\beta}$ with respect to $\beta - \beta_c$ with different S

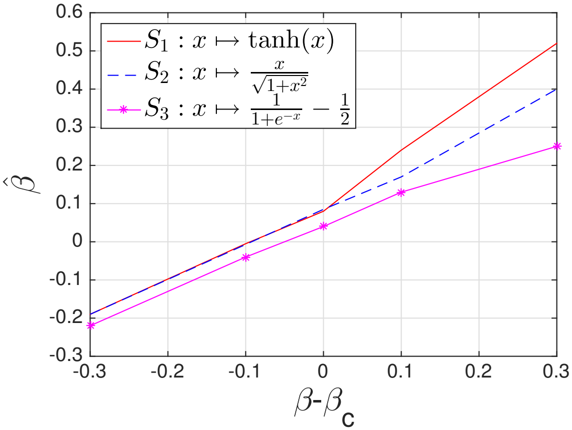




Evolution of $\hat{\beta}$ with respect to $\beta - \beta_c$ with different S, ε and u



Evolution of $\hat{\beta}$ with respect to $\beta - \beta_c$ with additive measurement noises



Outline

Problem statement

Qualitative estimator

Numerical simulations

Conclusion

Conclusion

- Concept of qualitative estimation
- Design of qualitative parameter estimator
  ▶ Provide online information about the actual behavior of the system
  ▶ Convergence guarantees

Perspectives

- Higher-dimensional models
- Qualitative parameter estimator design
  ▶ Without knowledge of the input and the slow variable

References: arXiv 2016, submitted to 20th IFAC World Congress, journal version in preparation.

Thank you for your attention