# SAMPLED OUTPUT OBSERVER FOR A CLASS OF UNCERTAIN NONLINEAR SYSTEMS APPLICATION TO BIOREACTORS

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## LAYOUT

- Motivation
- Quick overview
- The problem formulation
- The continuous-discrete time observer
- Illustrative example
- Conclusion

## MOTIVATION

*Design a continuous-discrete time observer*
*for a class of systems that are observable for any input*
*using the high gain concept*
*together with a constructive Lyapunov approach*

$\downarrow$

*implementation and calibration simplicity purposes*
*maximum allowable value for the sampling partition diameter*
*ability to alleviate the performances*

# A QUICK OVERVIEW

- F. Deza, E. Busvelle, J.P. Gauthier, D. Rakotopara (SCL, 1992)
- M. Nadri and H. Hammouri and C.M. Astorga Zaragoza (EJC, 2004)
- M. Nadri, H. Hammouri, R.M. Grajales (IEEE-TAC, 2013)
- I. Karafyllis and C. Kravaris (IEEE-TAC, 2009)
- T. Raff, M. Kögel and F. Allgöwer (ACC, 2008)
- M. Farza, M. M'Saad, M.L. Fall, O. Gehen, E. Pigeon, K. Busawon (IEEE-TAC, 2014)
- F. Mazenc, T.N. Dinh (Automatica, 2014)
- F. Mazenc, V. Andrieu, M. Malisoff (Automatica 2015)

• Few works dealing with continuous-discrete time observers for uncertain nonlinear systems

# PROBLEM FORMULATION

*To design a continuous-discrete time observer*
*from an appropriate redesign*
*of a continuous time high gain observer*
*for a class of MIMO nonlinear systems*

$\downarrow$

*M. Farza, M. M'Saad and L. Rossignol (Automatica, 2004)*

## Problem formulation

- Class of considered systems:

$$(\Sigma) \quad \begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t), x(t)) + B\varepsilon(t) \\ y(t_k) = Cx(t_k) = x^1(t_k) \end{cases}$$

with

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^{q-1} \\ x^q \end{pmatrix} \in \mathbb{R}^n, \quad \varphi(u,x) = \begin{pmatrix} \varphi^1(u, x^1) \\ \varphi^2(u, x^1, x^2) \\ \vdots \\ \varphi^{q-1}(u, x^1, \ldots, x^{q-1}) \\ \varphi^q(u, x) \end{pmatrix}$$

$$A = \begin{pmatrix} 0_p & I_p & & 0_p \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & I_p \\ 0_p & 0_p & \dots & 0_p \end{pmatrix}, \quad B = \begin{pmatrix} 0_p \\ \vdots \\ 0_p \\ I_p \end{pmatrix}, \quad C = \begin{pmatrix} I_p & 0_p & \dots & 0_p \end{pmatrix}$$

$x^i \in \mathbb{R}^p$, $i \in [1,\ q]$, $u(t) \in U$ a compact subset of $\mathbb{R}^m$, $y \in \mathbb{R}^p$ the system output available only at the sampling instants that satisfy $0 \leq t_0 < \dots < t_k < t_{k+1} < \dots$ with time-varying sampling intervals $\tau_k = t_{k+1} - t_k$ and $\lim\limits_{k \to \infty} t_k = +\infty$.

• The function $\varepsilon : \mathbb{R}^+ \mapsto \varepsilon(t) \in \mathbb{R}^p$ is an unknown function representing the system uncertainties and may depend on the state, the inputs and uncertain parameters.

## Assumptions

**A1.** The functions $\varphi^i$ for $i \in [1,\ q]$ are globally Lipschitz with respect to $x$ uniformly in $u$, i.e. there exists $L > 0$ such that for all $u \in U$ and for all $x, \bar{x} \in \mathbb{R}^n$, the following inequality holds for $i = 1, \ldots, q$:

$$\|\varphi^i (u, x) - \varphi^i (u, \bar{x})\| \leq L\|x - \bar{x}\|$$

**A2.** The unknown function $\varepsilon$ is bounded, i.e.

$$\exists \delta > 0;\ \forall t \geq 0 :\ \|\varepsilon(t)\| \leq \delta$$

Furthermore, one naturally assumes that the time intervals $\tau_k$'s are bounded away from zero by $\tau_m$ and are upperly bounded by the upper bound of the sampling partition diameter $\tau_M$, i.e.

$$0 < \tau_m \leq \tau_k = t_{k+1} - t_k \leq \tau_M,\quad \forall k \geq 0$$

# The high gain observer design with continuous outputs

$$(\mathcal{O}C) \quad \dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta\Delta_\theta^{-1}K(C\hat{x}(t) - y(t))$$

$$\text{with} \quad \Delta_\theta = diag\left(I_p, \frac{1}{\theta}I_p, \ldots, \frac{1}{\theta^{q-1}}I_p\right)$$

where $\hat{x} \in \mathbb{R}^n$ , $K \in \mathbb{R}^{n \times p}$ is such that $\bar{A} \stackrel{\Delta}{=} A - KC$ is an Hurwitz matrix and $\theta \geq 1$ is a scalar design parameter.

**Theorem:** *Consider system $(\Sigma)$ subject to assumptions* **A1** *and* **A2** *together with the observer $(\mathcal{O}C)$. Then,*

$$\exists\theta_0 > 0;\ \exists\lambda > 0;\ \forall\theta \geq \theta_0; \exists\mu_\theta > 0; \forall u \in U; \forall\hat{x}(0) \in I\!R^n;\ \textit{we have}$$

$$\|\hat{x}(t) - x(t)\| \leq \lambda\theta^{q-1}e^{-\mu_\theta t}\|\hat{x}(0) - x(0)\| + M_\theta\delta$$

*where $\delta$ is the upper bound of unknown function $\|\varepsilon\|$. Moreover, one has:* $\lim\limits_{\theta \to \infty} \mu_\theta = +\infty$ *and* $\lim\limits_{\theta \to \infty} M_\theta = 0.$

# The impulsive continuous-discrete time observer

$$
\begin{aligned}
\dot{\hat{x}}(t) &= A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) \\
&\quad -\theta \Delta_\theta^{\ \ 1} K e^{\ \ \theta K^{1}(t\ \ \ t_k)} (C\hat{x}(t_k) - y(t_k)),\ t \in [t_k,\ t_{k+1}[
\end{aligned}
$$

**Theorem:** *Consider the system* $(\Sigma)$) *subject to Assumption* **A1** *and* **A2**. *Then, there exists* $\lambda > 0$, *for every* $u \in U$, *for every* $\theta > \theta_0$, *there exists* $\chi_\theta > 0$, $\eta_\theta(\tau_M) > 0$, $N_\theta(\tau_m, \tau_M) > 0$ *such that if the upper bound of the sampling partition diameter* $\tau_M$ *is chosen such that* $\tau_M < \chi_\theta$, *then for every* $\hat{x}(0) \in I\!R^n$, *we have:*

$$
\|\hat{x}(t) - x(t)\| \leq \lambda \theta^{q\ \ 1} e^{\ \ \eta_\theta(\tau_M)t} \|\hat{x}(0) - x(0)\| + N_\theta(\tau_m, \tau_M)\delta
$$

*where* $\delta$ *is the upper bound of* $\|\varepsilon\|$ *and the parameters* $\lambda, \theta_0$ *are those of the continuous output case while* $\tau_m$ *and* $\tau_M$ *are the lower and upper bounds of the sampling partition diameter respectively.*

# Remark

- Continuous outputs:

$$\|\hat{x}(t) - x(t)\| \leq \lambda\theta^{q-1} e^{-\mu_\theta t}\|\hat{x}(0) - x(0)\| + M_\theta\delta$$

- Sampled outputs:

$$\|\hat{x}(t) - x(t)\| \leq \lambda\theta^{q-1} e^{-\eta_\theta(\tau_M)t}\|\hat{x}(0) - x(0)\| + N_\theta(\tau_m, \tau_M)\delta$$

- Constant sampling period $\tau_M = \tau_m = T_s$:

  - $T_s \mapsto \eta_\theta(T_s)$ is a non increasing function of $T_s$ with

$$\lim_{T_s \to 0} \eta_\theta(T_s) = \mu_\theta$$

  - $T_s \mapsto N_\theta(T_s)$ is an increasing function of $T_s$ with

$$\lim_{T_s \to 0} N_\theta(T_s) = M_\theta$$

## Continuous-discrete time observer

- Sketch of the proof:

Let $\tilde{x}(t) = \hat{x}(t) - x(t)$, $\bar{x} = \Delta_\theta \tilde{x}$, $\Phi(u, \hat{x}, x) = \varphi(u, \hat{x}) - \varphi(u, x)$ :

$$\dot{\bar{x}} = \underbrace{\theta \bar{A}\bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) - \frac{1}{\theta^{q \quad 1}} B\varepsilon(t)}_{\text{as in the continuous output case}} + \underbrace{\theta K z}_{\text{resulting from the sampled outputs}}$$

$$\text{with} \quad z(t) = \bar{x}^1(t) - e^{\quad \theta K^1 (t \quad t_k)} \bar{x}^1(t_k) \quad (z(t_k) = 0)$$

$$\Longrightarrow \dot{z}(t) = \dot{\bar{x}}^1(t) + K^1 \theta e^{\quad \theta K^1 (t \quad t_k)} \bar{x}^1(t_k) = \theta \bar{x}^2 + \Phi^1(u, \hat{x}^1, x^1)$$

$$\Longrightarrow \|z(t)\| \leq (\theta + L) \int_{t_k}^{t} \|\bar{x}(s)\| ds$$

# Sketch of the proof

- $V(\bar{x}) = \bar{x}^T P \bar{x}$ with $P\bar{A} + \bar{A}^T P \leq -2\mu I_n$:

$$\dot{V}(\bar{x}(t)) \leq \underbrace{-\frac{\mu\theta}{\lambda_M} V(\bar{x}(t)) + \frac{2\sqrt{\lambda_M}}{\theta^{q\ \ 1}} \delta \sqrt{V(\bar{x})}}_{\text{as in the continuous output case}}$$

$$\underbrace{+2\theta \sqrt{\frac{\lambda_M}{\lambda_m}} \|K\| (\theta + L) \sqrt{V(\bar{x}(t))} \int_{t_k}^{t} \sqrt{V(\bar{x}(s))} ds}_{\text{resulting from the sampled outputs}}$$

or equivalently

$$\frac{d}{dt}\sqrt{V(\bar{x}(t))} \leq -\underbrace{\frac{\mu\theta}{2\lambda_M}}_{a_\theta} \sqrt{V(\bar{x}(t))} + \underbrace{\theta\sqrt{\sigma}\|K\|(L+\theta)}_{b_\theta} \int_{t_k}^{t} \sqrt{V(\bar{x}(s))} ds + \underbrace{\frac{\sqrt{\lambda_M}}{\theta^{q\ \ 1}} \delta}_{c_\theta}$$

# A technical lemma

*Consider a differentiable function* $v : t \in R^+ \mapsto v(t) \in R^+$ *satisfying the following inequality:*

$$\dot{v}(t) \leq -av(t) + b\int_{t_k}^{t} v(s)ds + c \quad \forall t \in [t_k, t_{k+1}[ \text{ with } k \in I\!N,\ t_0 \geq 0$$

*where* $0 < \tau_m \leq \tau_k = t_{k+1} - t_k \leq \tau_M < +\infty$ *and* $a, b$ *and* $c$ *are positive reals satisfying*

$$\frac{b\tau_M}{a} < 1.$$

*Then, the function* $v$ *satisfies*

$$\begin{array}{rcl} v(t) & \leq & e^{\ \ \eta(t \quad t_0)}v(t_0) + c\tau_M\left(1 + \dfrac{1}{1 - e^{\ \ \eta\tau_m}}\right) \\ \text{with} \quad 0 < \eta & = & (a - b\tau_M)\, e^{\ \ a\tau_M} \end{array}$$

## Predictor output form

- The observer as a hybrid system:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta\Delta_\theta{}^{1}K(C\hat{x}(t) - \omega(t))$$

with $w(t)$ prediction of the system output governed by

$$\begin{cases} \dot{\omega}(t) = \hat{x}^2(t) + \varphi^1(u(t), \hat{x}^1(t)) \; for \; t \in [t_k, t_{k+1}[ \\ \omega(t_k) = Cx(t_k) = y(t_k) \end{cases}$$

- Set $\xi(t) \stackrel{\Delta}{=} C\hat{x}(t) - \omega(t) = \hat{x}^1(t) - \omega(t) \Longrightarrow \dot{\xi}(t) = -\theta K^1\xi(t)$

$\Longrightarrow \xi(t) = e^{\;\theta K^1(t \;\; t_k)}\xi(t_k)$ .

Since $\omega(t_k) = y(t_k) \Longrightarrow \xi(t_k) = \hat{x}^1(t_k) - \omega(t_k) \; = \; C\hat{x}(t_k) - y(t_k)$.

$\Longrightarrow \xi(t) = e^{\;\theta K^1(t \;\; t_k)}\,(C\hat{x}(t_k) - y(t_k))$ **Impulsive form of the observer**

## A relevant remark

A continuous-discrete time observer candidate can be given by the following more general form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta\Delta_\theta{}^{1} KF(t)(C\hat{x}(t_k) - y(t_k))$$

where the matrix function $F(t) \in \mathbb{R}^{p\ \ p}$ satisfies $F(t_k) = I_p$ and is naturally specified to provide the the maximum allowable value for the sampling partition diameter $\tau_M$.

Proceeding as in the involved convergence analysis, one obtains

$$\dot{\bar{x}} = \theta A\bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) - \theta KF(t)C\bar{x}(t_k)$$

And adding and subtracting $\theta KC\bar{x}(t)$ to the right side of the above equation yields

$$\dot{\bar{x}} = \theta \bar{A}\bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) + \theta Kz$$

with

$$\bar{A} = A - KC$$

$$z(t) = C\bar{x}(t) - F(t)C\bar{x}(t_k) = \bar{x}^1(t) - F(t)\bar{x}^1(t_k), \quad z(t_k) = 0$$

The time derivative of $z$ can be hence expressed as follows

$$\dot{z}(t) = \theta\bar{x}^2(t) + \Phi^1(u(t), \hat{x}(t), x(t)) - (\theta K^1 F(t) + \dot{F}(t))\bar{x}^1(t_k)$$

$$\downarrow$$

If one chooses the matrix $F(t)$ equal to the solution of the following matrix ordinary differential equation

$$\theta K^1 F(t) + \dot{F}(t) = 0 \text{ with } F(t_k) = I_p$$

then the equation of the time derivative of $z$ becomes

$$\dot{z}(t) = \theta\bar{x}^2(t) + \Phi^1(u, \hat{x}, x)$$

It is clear that the unique solution of the matrix ordinary differential equation

$$\theta K^{1} F(t) + \dot{F}(t) = 0 \text{ with } F(t_k) = I_p$$

is $F(t) = e^{\ \theta K^{1}(t \quad t_k)}$: one hence recover the proposed observer. It should be emphasized that other choices of $F(t)$ are possible, e.g. $F(t) = I_p$, but they give rise to lower values of the maximum allowable value for the sampling partition diameter.

# Example: a typical bioreactor

• A biomass $X$ is growing by consuming a substrate $S$ in a continous stirred tank biorecator with an input substrate concentration $S_{in}$ and a dilution rate $D$:

$$\begin{cases} \dot{X} &= r - DX \\ \dot{S} &= -kr + D(S_{in} - S) \\ y(t_k) &= X(t_k) \end{cases}$$

The reaction rate $r(t)$ is generally a very complex function of the states and the environmental factors. The objective is to consider this key variable as a time-varying parameter. The objective is to estimate this rate from the measurements of the biomass concentration available at sampling instants.

• System considered for the observer design

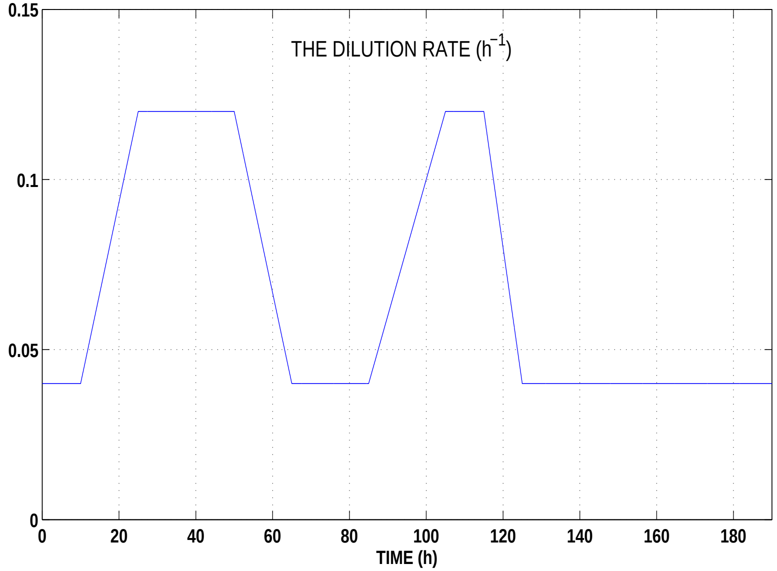
$$\begin{cases} \dot{X}(t) &= r(t) - D(t)X(t) \\ \dot{r}(t) &= \varepsilon(t) \\ y(kT_s) &= X(kT_s) \end{cases}$$

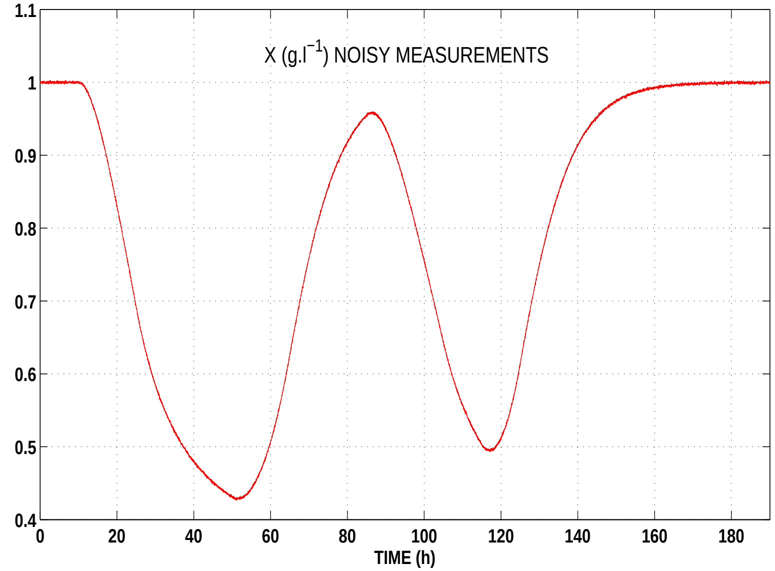
• An expression for the reaction $r(t)$ is needed to generate the pseudo measurements of $X$:

$$r = \frac{\mu^\star SX}{k_C X + S}$$

• **The above expression is unknown by the observer.**

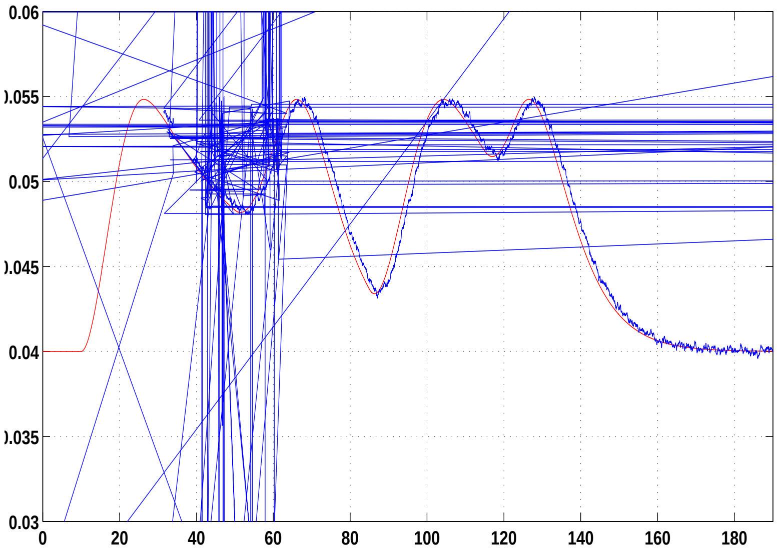
# Simulation results

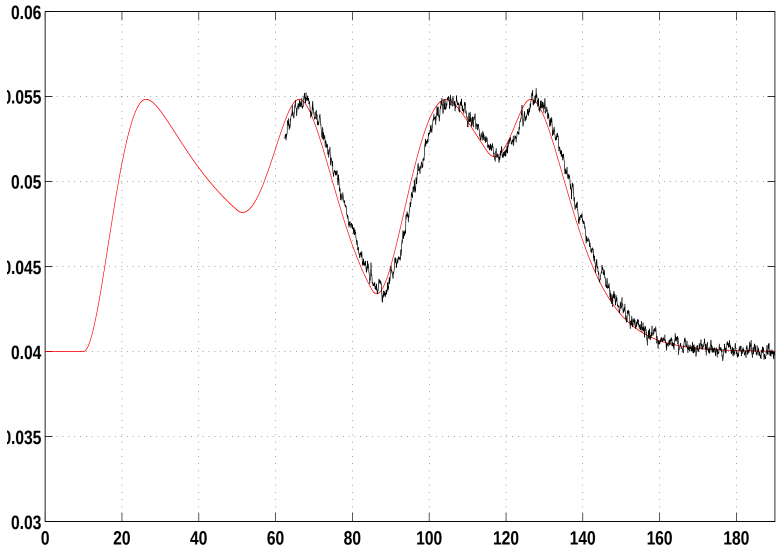


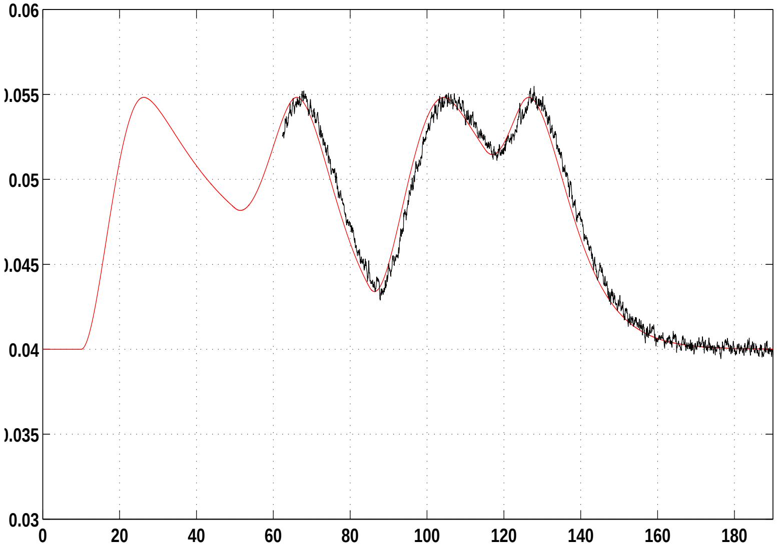
# Simulation results

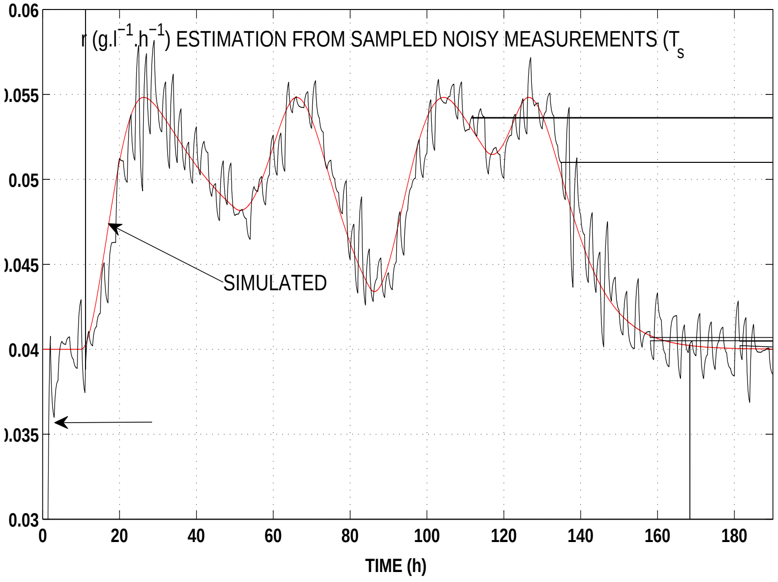
- $\theta = 1.5$



# Simulation results



## CONCLUSION

- A continuous-discrete time observer has been proposed for a triangular MIMO systems bearing in mind the implementation and calibration simplicity
- A constructive Lyapunov approach for the convergence analysis to provide useful expressions for the observer gain as well as the maximum allowable value of the sampling partition diameter
- Suitable interpretation and a general candidate of the impulsive observer form
- Simulation results throughout an illustrative example
- Extension to time delay systems, adaptive observers and observers with unknown inputs