

SAMPLED OUTPUT OBSERVER FOR A CLASS

OF UNCERTAIN NONLINEAR SYSTEMS

APPLICATION TO BIOREACTORS

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LAYOUT

- Motivation
- Quick overview
- The problem formulation
- The continuous-discrete time observer
- Illustrative example
- Conclusion

MOTIVATION

*Design a continuous-discrete time observer
for a class of systems that are observable for any input
using the high gain concept
together with a constructive Lyapunov approach*



*implementation and calibration simplicity purposes
maximum allowable value for the sampling partition diameter
ability to alleviate the performances*

A QUICK OVERVIEW

- F. Deza, E. Busvelle, J.P. Gauthier, D. Rakotopara (SCL, 1992)
- M. Nadri and H. Hammouri and C.M. Astorga Zaragoza (EJC, 2004)
- M. Nadri, H. Hammouri, R.M. Grajales (IEEE-TAC, 2013)
- I. Karafyllis and C. Kravaris (IEEE-TAC, 2009)
- T. Raff, M. Kögel and F. Allgöwer (ACC, 2008)
- M. Farza, M. M'Saad, M.L. Fall, O. Gehen, E. Pigeon, K. Busawon (IEEE-TAC, 2014)
- F. Mazenc, T.N. Dinh (Automatica, 2014)
- F. Mazenc, V. Andrieu, M. Malisoff (Automatica 2015)
- Few works dealing with continuous-discrete time observers for uncertain nonlinear systems

PROBLEM FORMULATION

*To design a continuous-discrete time observer
from an appropriate redesign
of a continuous time high gain observer
for a class of MIMO nonlinear systems*



M. Farza, M. M'Saad and L. Rossignol (Automatica, 2004)

Problem formulation

- Class of considered systems:

$$(\Sigma) \quad \begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t), x(t)) + B\varepsilon(t) \\ y(t_k) = Cx(t_k) = x^1(t_k) \end{cases}$$

with

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^{q-1} \\ x^q \end{pmatrix} \in \mathbb{R}^n, \quad \varphi(u, x) = \begin{pmatrix} \varphi^1(u, x^1) \\ \varphi^2(u, x^1, x^2) \\ \vdots \\ \varphi^{q-1}(u, x^1, \dots, x^{q-1}) \\ \varphi^q(u, x) \end{pmatrix}$$

$$A = \begin{pmatrix} 0_p & I_p & & 0_p \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & I_p \\ 0_p & 0_p & \dots & 0_p \end{pmatrix}, \quad B = \begin{pmatrix} 0_p \\ \vdots \\ 0_p \\ I_p \end{pmatrix}, \quad C = \begin{pmatrix} I_p & 0_p & \dots & 0_p \end{pmatrix}$$

$x^i \in \mathbb{R}^p$, $i \in [1, q]$, $u(t) \in U$ a compact subset of \mathbb{R}^m , $y \in \mathbb{R}^p$ the system output available only at the sampling instants that satisfy

$0 \leq t_0 < \dots < t_k < t_{k+1} < \dots$ with time-varying sampling intervals

$\tau_k = t_{k+1} - t_k$ and $\lim_{k \rightarrow \infty} t_k = +\infty$.

- The function $\varepsilon : \mathbb{R}^+ \mapsto \varepsilon(t) \in \mathbb{R}^p$ is an unknown function representing the system uncertainties and may depend on the state, the inputs and uncertain parameters.

Assumptions

A1. The functions φ^i for $i \in [1, q]$ are globally Lipschitz with respect to x uniformly in u , i.e. there exists $L > 0$ such that for all $u \in U$ and for all $x, \bar{x} \in \mathbb{R}^n$, the following inequality holds for $i = 1, \dots, q$:

$$\|\varphi^i(u, x) - \varphi^i(u, \bar{x})\| \leq L\|x - \bar{x}\|$$

A2. The unknown function ε is bounded, i.e.

$$\exists \delta > 0; \forall t \geq 0 : \|\varepsilon(t)\| \leq \delta$$

Furthermore, one naturally assumes that the time intervals τ_k 's are bounded away from zero by τ_m and are upperly bounded by the upper bound of the sampling partition diameter τ_M , i.e.

$$0 < \tau_m \leq \tau_k = t_{k+1} - t_k \leq \tau_M, \quad \forall k \geq 0$$

The high gain observer design with continuous outputs

$$(OC) \quad \dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta \Delta_\theta^{-1} K (C\hat{x}(t) - y(t))$$

$$\text{with } \Delta_\theta = \text{diag} \left(I_p, \frac{1}{\theta} I_p, \dots, \frac{1}{\theta^{q-1}} I_p \right)$$

where $\hat{x} \in \mathbb{R}^n$, $K \in \mathbb{R}^{n \times p}$ is such that $\bar{A} \triangleq A - KC$ is an Hurwitz matrix and $\theta \geq 1$ is a scalar design parameter.

Theorem: Consider system (Σ) subject to assumptions **A1** and **A2** together with the observer (OC). Then,

$\exists \theta_0 > 0; \exists \lambda > 0; \forall \theta \geq \theta_0; \exists \mu_\theta > 0; \forall u \in U; \forall \hat{x}(0) \in \mathbb{R}^n;$ we have

$$\|\hat{x}(t) - x(t)\| \leq \lambda \theta^{q-1} e^{-\mu_\theta t} \|\hat{x}(0) - x(0)\| + M_\theta \delta$$

where δ is the upper bound of unknown function $\|\varepsilon\|$. Moreover, one has:

$$\lim_{\theta \rightarrow \infty} \mu_\theta = +\infty \text{ and } \lim_{\theta \rightarrow \infty} M_\theta = 0.$$

The impulsive continuous-discrete time observer

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) \\ &\quad - \theta \Delta_\theta^{-1} K e^{-\theta K^1(t-t_k)} (C\hat{x}(t_k) - y(t_k)), \quad t \in [t_k, t_{k+1}[\end{aligned}$$

Theorem: Consider the system (Σ) subject to Assumption **A1** and **A2**. Then, there exists $\lambda > 0$, for every $u \in U$, for every $\theta > \theta_0$, there exists $\chi_\theta > 0$, $\eta_\theta(\tau_M) > 0$, $N_\theta(\tau_m, \tau_M) > 0$ such that if the upper bound of the sampling partition diameter τ_M is chosen such that $\tau_M < \chi_\theta$, then for every $\hat{x}(0) \in \mathbb{R}^n$, we have:

$$\|\hat{x}(t) - x(t)\| \leq \lambda \theta^{q-1} e^{-\eta_\theta(\tau_M)t} \|\hat{x}(0) - x(0)\| + N_\theta(\tau_m, \tau_M) \delta$$

where δ is the upper bound of $\|\varepsilon\|$ and the parameters λ, θ_0 are those of the continuous output case while τ_m and τ_M are the lower and upper bounds of the sampling partition diameter respectively.

Remark

- Continuous outputs:

$$\|\hat{x}(t) - x(t)\| \leq \lambda\theta^{q-1}e^{-\mu_\theta t} \|\hat{x}(0) - x(0)\| + M_\theta\delta$$

- Sampled outputs:

$$\|\hat{x}(t) - x(t)\| \leq \lambda\theta^{q-1}e^{-\eta_\theta(\tau_M)t} \|\hat{x}(0) - x(0)\| + N_\theta(\tau_m, \tau_M)\delta$$

- Constant sampling period $\tau_M = \tau_m = T_s$:

- $T_s \mapsto \eta_\theta(T_s)$ is a non increasing function of T_s with

$$\lim_{T_s \rightarrow 0} \eta_\theta(T_s) = \mu_\theta$$

- $T_s \mapsto N_\theta(T_s)$ is an increasing function of T_s with

$$\lim_{T_s \rightarrow 0} N_\theta(T_s) = M_\theta$$

Continuous-discrete time observer

- Sketch of the proof:

Let $\tilde{x}(t) = \hat{x}(t) - x(t)$, $\bar{x} = \Delta_\theta \tilde{x}$, $\Phi(u, \hat{x}, x) = \varphi(u, \hat{x}) - \varphi(u, x)$:

$$\dot{\tilde{x}} = \underbrace{\theta \bar{A} \bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) - \frac{1}{\theta^{q-1}} B \varepsilon(t)}_{\text{as in the continuous output case}} + \underbrace{\theta K z}_{\text{resulting from the sampled outputs}}$$

$$\text{with } z(t) = \bar{x}^1(t) - e^{-\theta K^1(t-t_k)} \bar{x}^1(t_k) \quad (z(t_k) = 0)$$

$$\implies \dot{z}(t) = \dot{\bar{x}}^1(t) + K^1 \theta e^{-\theta K^1(t-t_k)} \bar{x}^1(t_k) = \theta \bar{x}^2 + \Phi^1(u, \hat{x}^1, x^1)$$

$$\implies \|z(t)\| \leq (\theta + L) \int_{t_k}^t \|\bar{x}(s)\| ds$$

Sketch of the proof

- $V(\bar{x}) = \bar{x}^T P \bar{x}$ with $P\bar{A} + \bar{A}^T P \leq -2\mu I_n$:

$$\dot{V}(\bar{x}(t)) \leq \underbrace{-\frac{\mu\theta}{\lambda_M} V(\bar{x}(t)) + \frac{2\sqrt{\lambda_M}}{\theta^{q-1}} \delta \sqrt{V(\bar{x})}}_{\text{as in the continuous output case}}$$

as in the continuous output case

$$+ \underbrace{2\theta \sqrt{\frac{\lambda_M}{\lambda_m}} \|K\| (\theta + L) \sqrt{V(\bar{x}(t))} \int_{t_k}^t \sqrt{V(\bar{x}(s))} ds}_{\text{resulting from the sampled outputs}}$$

resulting from the sampled outputs

or equivalently

$$\frac{d}{dt} \sqrt{V(\bar{x}(t))} \leq - \underbrace{\frac{\mu\theta}{2\lambda_M}}_{a_\theta} \sqrt{V(\bar{x}(t))} + \underbrace{\theta\sqrt{\sigma}\|K\|(L+\theta)}_{b_\theta} \int_{t_k}^t \sqrt{V(\bar{x}(s))} ds + \underbrace{\frac{\sqrt{\lambda_M}}{\theta^{q-1}} \delta}_{c_\theta}$$

A technical lemma

Consider a differentiable function $v : t \in \mathbb{R}^+ \mapsto v(t) \in \mathbb{R}^+$ satisfying the following inequality:

$$\dot{v}(t) \leq -av(t) + b \int_{t_k}^t v(s)ds + c \quad \forall t \in [t_k, t_{k+1}[\text{ with } k \in \mathbb{N}, t_0 \geq 0$$

where $0 < \tau_m \leq \tau_k = t_{k+1} - t_k \leq \tau_M < +\infty$ and a, b and c are positive reals satisfying

$$\frac{b\tau_M}{a} < 1.$$

Then, the function v satisfies

$$v(t) \leq e^{-\eta(t-t_0)}v(t_0) + c\tau_M \left(1 + \frac{1}{1 - e^{-\eta\tau_m}} \right)$$

$$\text{with } 0 < \eta = (a - b\tau_M) e^{-a\tau_M}$$

Predictor output form

- The observer as a hybrid system:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta \Delta_\theta^{-1} K (C\hat{x}(t) - \omega(t))$$

with $w(t)$ prediction of the system output governed by

$$\begin{cases} \dot{\omega}(t) = \hat{x}^2(t) + \varphi^1(u(t), \hat{x}^1(t)) \text{ for } t \in [t_k, t_{k+1}[\\ \omega(t_k) = Cx(t_k) = y(t_k) \end{cases}$$

- Set $\xi(t) \triangleq C\hat{x}(t) - \omega(t) = \hat{x}^1(t) - \omega(t) \implies \dot{\xi}(t) = -\theta K^1 \xi(t)$

$$\implies \xi(t) = e^{-\theta K^1 (t-t_k)} \xi(t_k) .$$

Since $\omega(t_k) = y(t_k) \implies \xi(t_k) = \hat{x}^1(t_k) - \omega(t_k) = C\hat{x}(t_k) - y(t_k)$.

$$\implies \xi(t) = e^{-\theta K^1 (t-t_k)} (C\hat{x}(t_k) - y(t_k)) \text{ Impulsive form of the observer}$$

A relevant remark

A continuous-discrete time observer candidate can be given by the following more general form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta \Delta_\theta^{-1} K F(t) (C\hat{x}(t_k) - y(t_k))$$

where the matrix function $F(t) \in \mathbb{R}^{p \times p}$ satisfies $F(t_k) = I_p$ and is naturally specified to provide the the maximum allowable value for the sampling partition diameter τ_M .

Proceeding as in the involved convergence analysis, one obtains

$$\dot{\bar{x}} = \theta A\bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) - \theta K F(t) C\bar{x}(t_k)$$

And adding and subtracting $\theta K C\bar{x}(t)$ to the right side of the above equation yields

$$\dot{\bar{x}} = \theta \bar{A}\bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) + \theta K z$$

with

$$\bar{A} = A - KC$$

$$z(t) = C\bar{x}(t) - F(t)C\bar{x}(t_k) = \bar{x}^1(t) - F(t)\bar{x}^1(t_k), \quad z(t_k) = 0$$

The time derivative of z can be hence expressed as follows

$$\dot{z}(t) = \theta\bar{x}^2(t) + \Phi^1(u(t), \hat{x}(t), x(t)) - (\theta K^1 F(t) + \dot{F}(t))\bar{x}^1(t_k)$$

↓

If one chooses the matrix $F(t)$ equal to the solution of the following matrix ordinary differential equation

$$\theta K^1 F(t) + \dot{F}(t) = 0 \text{ with } F(t_k) = I_p$$

then the equation of the time derivative of z becomes

$$\dot{z}(t) = \theta\bar{x}^2(t) + \Phi^1(u, \hat{x}, x)$$

It is clear that the unique solution of the matrix ordinary differential equation

$$\theta K^1 F(t) + \dot{F}(t) = 0 \text{ with } F(t_k) = I_p$$

is $F(t) = e^{-\theta K^1(t-t_k)}$: one hence recover the proposed observer. It should be emphasized that other choices of $F(t)$ are possible, e.g. $F(t) = I_p$, but they give rise to lower values of the maximum allowable value for the sampling partition diameter.

Example: a typical bioreactor

- A biomass X is growing by consuming a substrate S in a continuous stirred tank bioreactor with an input substrate concentration S_{in} and a dilution rate D :

$$\begin{cases} \dot{X} &= r - DX \\ \dot{S} &= -kr + D(S_{in} - S) \\ y(t_k) &= X(t_k) \end{cases}$$

The reaction rate $r(t)$ is generally a very complex function of the states and the environmental factors. The objective is to consider this key variable as a time-varying parameter. The objective is to estimate this rate from the measurements of the biomass concentration available at sampling instants.

- System considered for the observer design

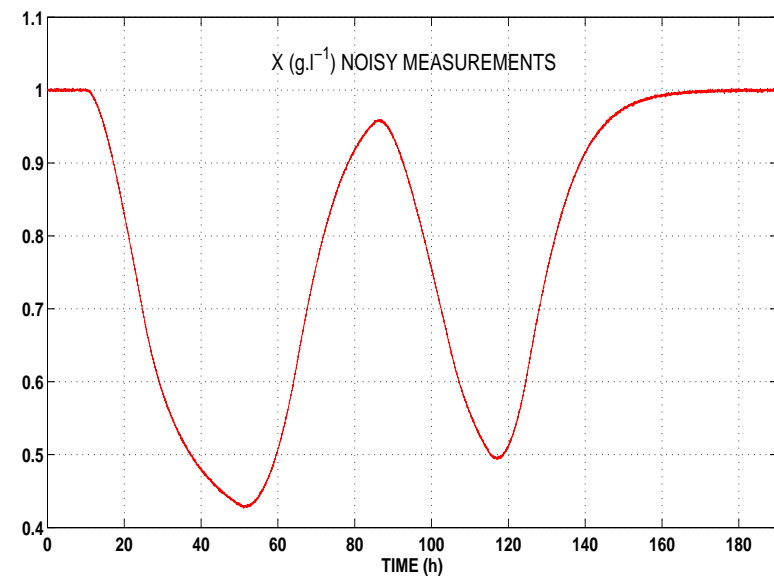
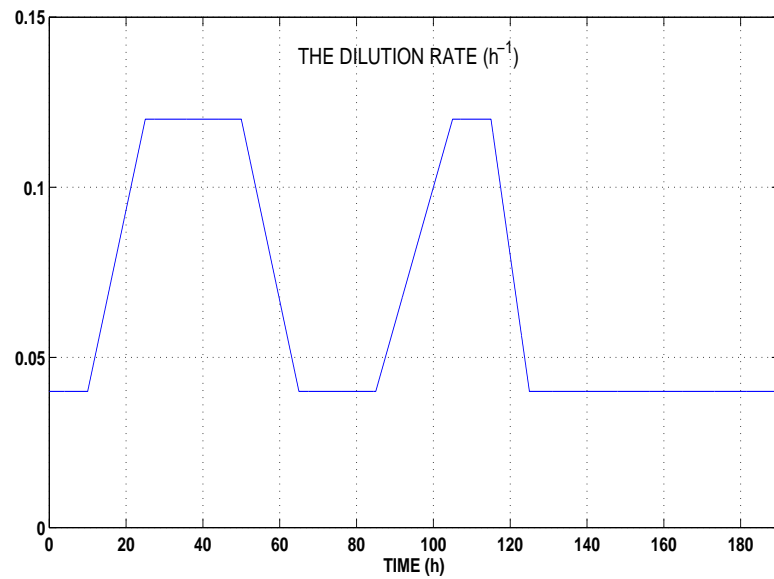
$$\begin{cases} \dot{X}(t) &= r(t) - D(t)X(t) \\ \dot{r}(t) &= \varepsilon(t) \\ y(kT_s) &= X(kT_s) \end{cases}$$

- An expression for the reaction $r(t)$ is needed to generate the pseudo measurements of X :

$$r = \frac{\mu^* SX}{k_C X + S}$$

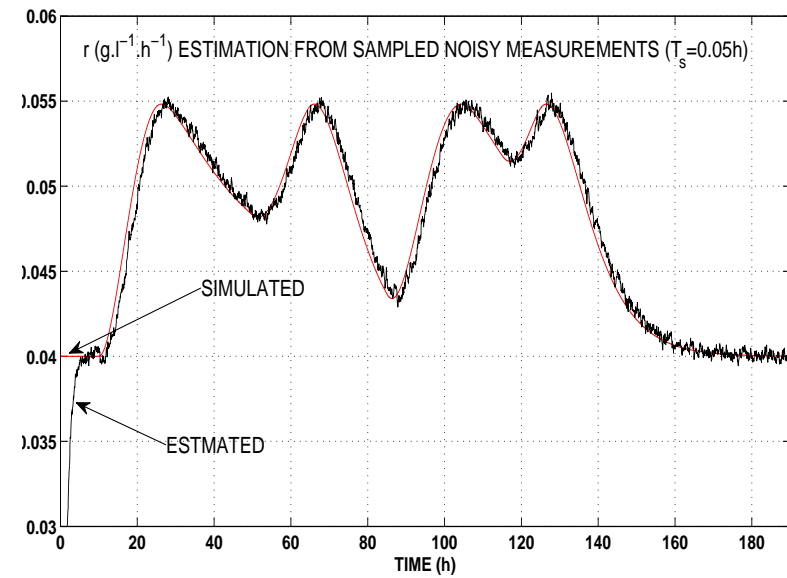
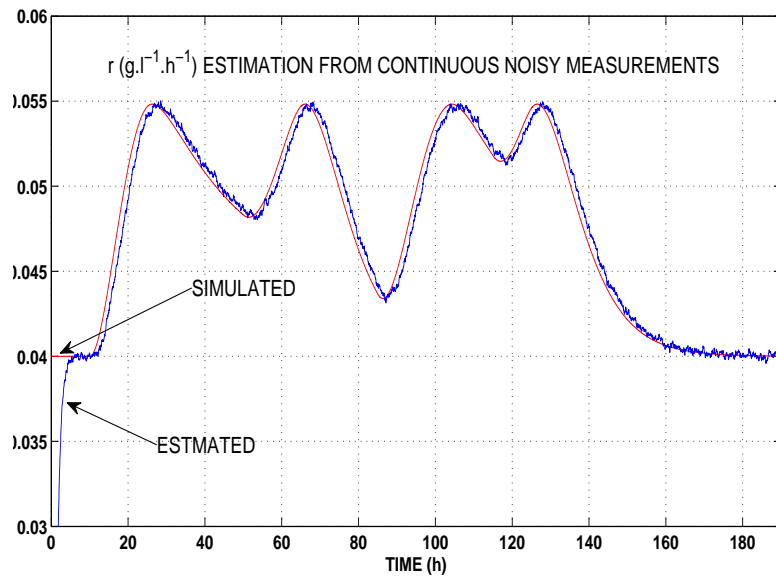
- **The above expression is unknown by the observer.**

Simulation results

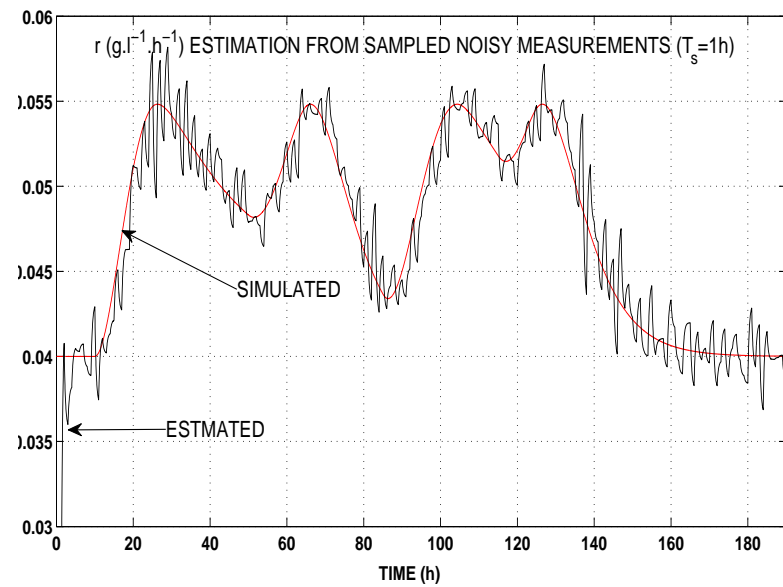
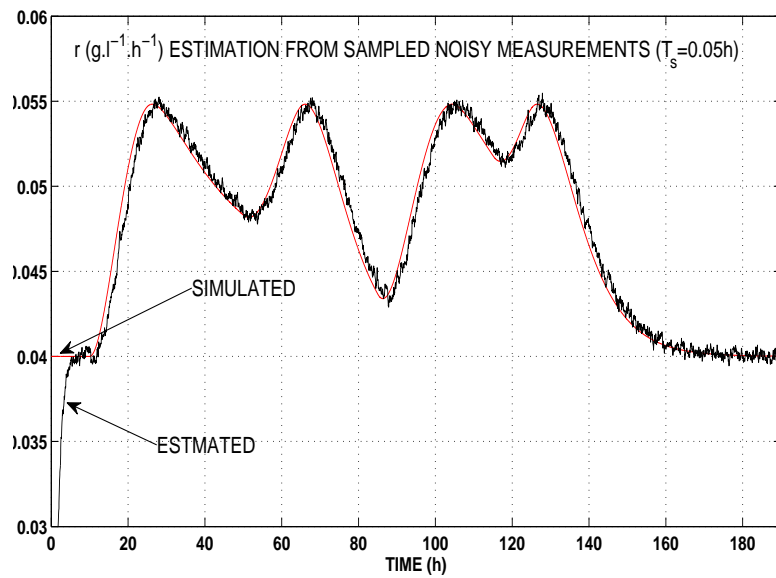


Simulation results

- $\theta = 1.5$



Simulation results



CONCLUSION

- A continuous-discrete time observer has been proposed for a triangular MIMO systems bearing in mind the implementation and calibration simplicity
- A constructive Lyapunov approach for the convergence analysis to provide useful expressions for the observer gain as well as the maximum allowable value of the sampling partition diameter
- Suitable interpretation and a general candidate of the impulsive observer form
- Simulation results throughout an illustrative example
- Extension to time delay systems, adaptive observers and observers with unknown inputs