Stability of the Kalman Filter for Continuous Time Output Error Systems

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Introduction

Classical linear system state estimators − Luenberger observer and Kalman filter − have similar structures.

- **Luenberger observer**: deterministic point of view, stability of the error dynamics arbitrarily tunable (observability condition).

- **Kalman filter**: stochastic point of view, optimal in the sense of minimum variance.

Stability of the (error dynamics) of the Kalman filter?

Introduction

- Classical stability result of the Kalman filter (Kalman 1963), assumes that the considered linear system is observable and controllable regarding the process noise.

- Here we consider process noise-free systems, which are obviously uncontrollable regarding the noise.

Motivations:
- Noise-free physical state equations
- Output-error system identification

A few words about output error system identification

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + 0$$
$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t)$$

Classical Prediction Error Method (PEM) is usually limited to **stable** Linear Time Invariant (LTI) systems, yet intermediate iterations may result in **unstable models** for weakly stable systems (typically unstable poles are projected into the stable region).

Considered Linear Time Varying (LTV) systems

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt$$
$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t)$$

cover also linear parameter varying (LPV) systems and state affine systems

$$dx(t) = A(t, u(t), y(t))x(t)dt + B(t, u(t), y(t))u(t)dt$$
$$dy(t) = C(t, u(t), y(t))x(t)dt + R^{\frac{1}{2}}(t)d\eta(t)$$

Classical stability results

System:

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + Q^{\frac{1}{2}}(t)d\omega(t)$$

$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t)$$

Assumptions:

- uniform observability
- uniform controllability

⇒

- Bounded Riccati equation
- Stable error equation

The controllability assumption refers to the process noise term $Q^{\frac{1}{2}}(t)d\omega(t)$

Kalman filter for output error systems

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt \;\mathbf{+\;0}$$

$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t)$$

Absence of process noise: the controllability condition in the classical results cannot be satisfied.

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + B(t)u(t)dt + K(t)(dy(t) - C(t)\hat{x}(t)dt)$$

$$K(t) = P(t)C^T(t)R^{-1}(t)$$

$$\frac{d}{dt}P(t) = A(t)P(t) + P(t)A^T(t) - P(t)C(t)^T R^{-1}(t)C(t)P(t)$$

$$\hat{x}(t_0) = x_0, \quad P(t_0) = P_0$$

Main new results

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt \; \mathbf{+ \; 0}$$

$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t)$$

Uniform observability ⇒

- Bounded Riccati equation
- Stable error equation

No controllability condition required!

Properties of the Riccati equation

If $A(t), B(t), C(t), R(t)$ are bounded and piecewise continuous, $P(t_0)$ is positive definite, and $[A(t), C(t)]$ is uniformly observable, then the solution of

$$\frac{d}{dt}P(t) = A(t)P(t) + P(t)A^T(t) - P(t)C(t)^T R^{-1}(t)C(t)P(t) + \mathbf{0}$$

Missing Q(t)

is positive definite and upper bounded.

Boundedness of the Kalman filter

The solution of the Riccati equation $P(t)$ is bounded, ……

so is the Kalman gain

$$K(t) = P(t)C^T(t)R^{-1}(t).$$

Properties of the Riccati equation

Proof hints: let $P(t) = \Omega^{-1}(t)$, then $\Omega(t)$ satisfies the (linear) Lyapunov equation

$$\frac{d\Omega(t)}{dt} + A^T(t)\Omega(t) + A(t)\Omega(t) = C^T(t)R^{-1}(t)C(t)$$

$$\Omega(t_0) = P_0^{-1}$$

$$\boxed{\begin{aligned}\Omega(t) = \Phi^T(t_0, t)\Omega(t_0)\Phi(t_0, t) \\ + \int_{t_0}^{t} \Phi^T(s, t)C^T(s)R^{-1}(s)C(s)\Phi(s, t)ds\end{aligned}}$$

The error dynamics of the Kalman filter

$$\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$$

Noise term

$$d\tilde{x}(t) = (A(t) - K(t)C(t))\tilde{x}(t)dt - K(t)R^{\frac{1}{2}}(t)d\eta(t)$$

The deterministic error dynamics

$$\frac{d}{dt}z(t) = (A(t) - K(t)C(t))z(t)$$

Asymptotic stability of the Kalman filter

If $A(t), B(t), C(t), R(t)$ are bounded and piecewise continuous, $P(t_0)$ is positive definite, and $[A(t), C(t)]$ is uniformly observable, then the deterministic error dynamics of the Kalman filter

$$\frac{d}{dt}z(t) = (A(t) - K(t)C(t))z(t)$$

is asymptotically stable.

The stability of $\dot{x}(t) = A(t)x(t)$ is not required.

Hints for the stability proof

For the deterministic error dynamics

$$\frac{d}{dt}z(t) = (A(t) - K(t)C(t))z(t)$$

define the "natural" Lyapunov function candidate

$$V(z(t),t) \triangleq z^T(t)P^{-1}(t)z(t)$$

In the classical case (uniform observability & controllability), *P*(*t*) has strictly positive **upper & lower** bounds → classical Lyapunov statibility analysis.

For output error systems, *P*(*t*) has no strictly positive lower bound (may tend to zero)!

Hints for the stability proof

Typically singular matrix

$$\frac{dV(z(t),t)}{dt} = -z^T(t)\boxed{C^T(t)R^{-1}(t)C(t)}z(t) \leq 0$$

→V(z(t),t) does not increase, but does it tend to zero?

In the classical case,

$$\frac{dV(z(t),t)}{dt} = -z^T(t)C^T(t)R^{-1}(t)C(t)z(t) - \boxed{z^T(t)P^{-1}(t)Q(t)P^{-1}(t)z(t)}$$

Missing for output error systems

Hints for the stability proof

The proof is based on the following lemma.

> If the pair [A(t), C(t)] is uniformly observable, then so is the pair [A(t)−K(t)C(t), C(t)] for any bounded K(t).

A classical result revisited in

Observability conservation by output feedback and observability Gramian bounds.

Zhang & Zhang, Automatica 60:38-42, 2015.

Exponential stability of the Kalman filter

Moreover, if $\dot{x}(t) = A(t)x(t)$ is exponentially stable,

or if $\dot{x}(t) = -A(t)x(t)$ is exponentially stable (anti-stable or strongly instable), then the error dynamics of the Kalman filter is exponentially stable.

Lyapunov stable output error systems

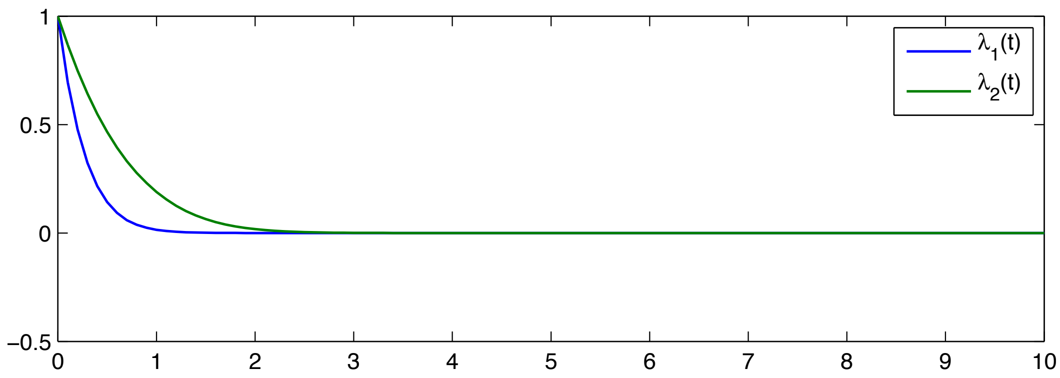
If $\dot{x}(t) = A(t)x(t)$ is Lyapunov stable, that is, the state transition matrix $\|\Phi(t, t_0)\|$ is bounded, then the error dynamics of the Kalman filter satisfies

$$\|z(t)\|^2 \leq \frac{\mu}{t - t_0} \|z(t_0)\|^2$$

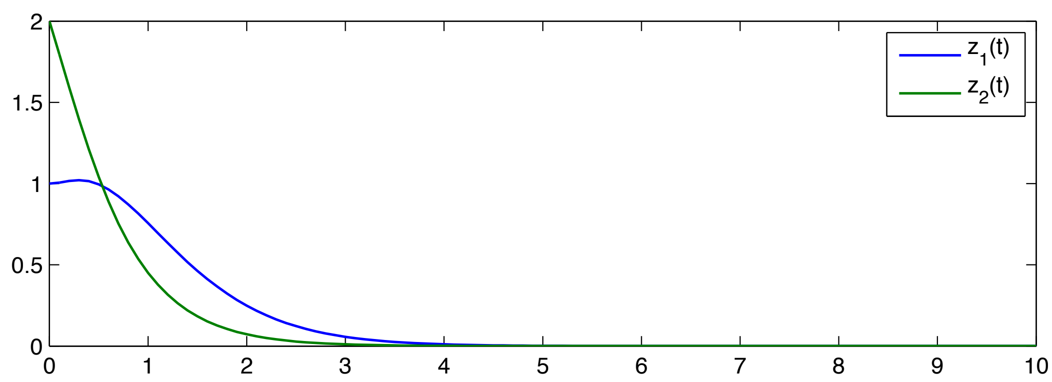
Numerical example 1

Exponentially stable

$$A(t) = \begin{bmatrix} e^{-t}-2 & 2-e^{-t} \\ 0 & e^{-t}-2 \end{bmatrix}, \quad C(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad R(t) = 1.$$



Eigenvalues of P(t)

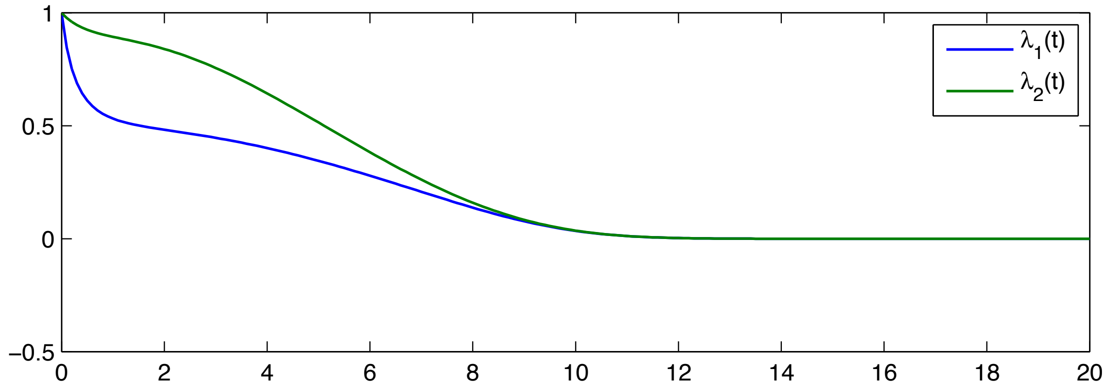


Error dynamics z(t)

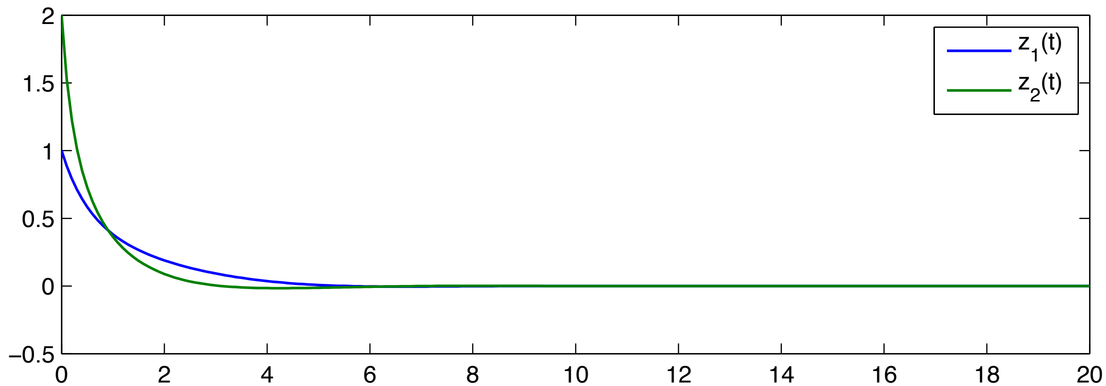
Numerical example 2

Lyapunov stable

$$A(t) = \begin{bmatrix} \cos(0.2t) & \sin(0.2t) \\ -\sin(0.2t) & \cos(0.2t) \end{bmatrix}, \quad C(t) = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}, \quad R(t) = I_2.$$



Eigenvalues of P(t)

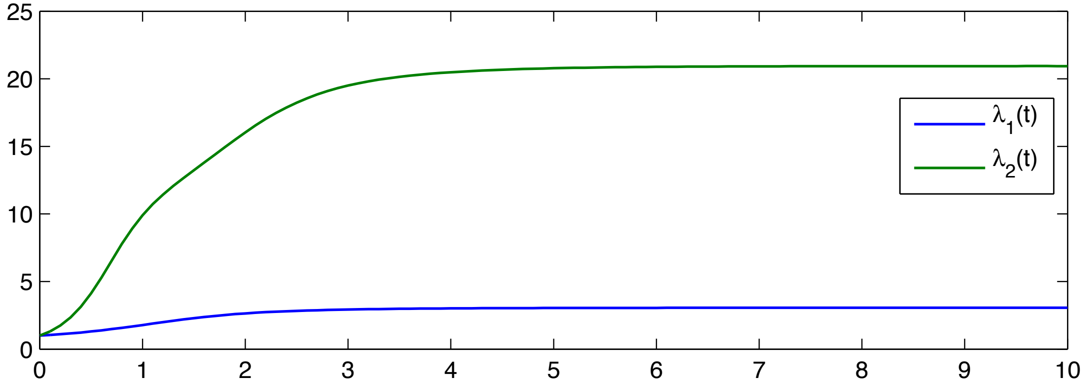


Error dynamics z(t)

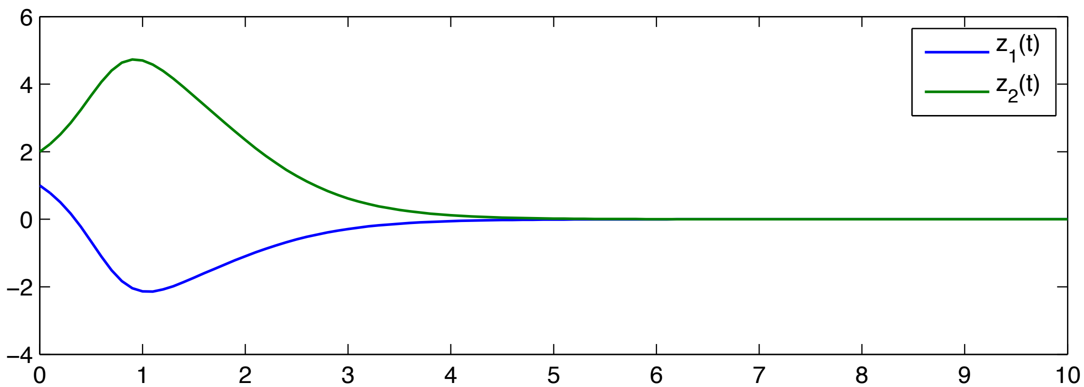
Numerical example 3

Unstable (anti-stable)

$$A(t) = \begin{bmatrix} 2-e^{-t} & e^{-t}-2 \\ 0 & 2-e^{-t} \end{bmatrix}, \quad C(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad R(t) = 1.$$



Eigenvalues of P(t)

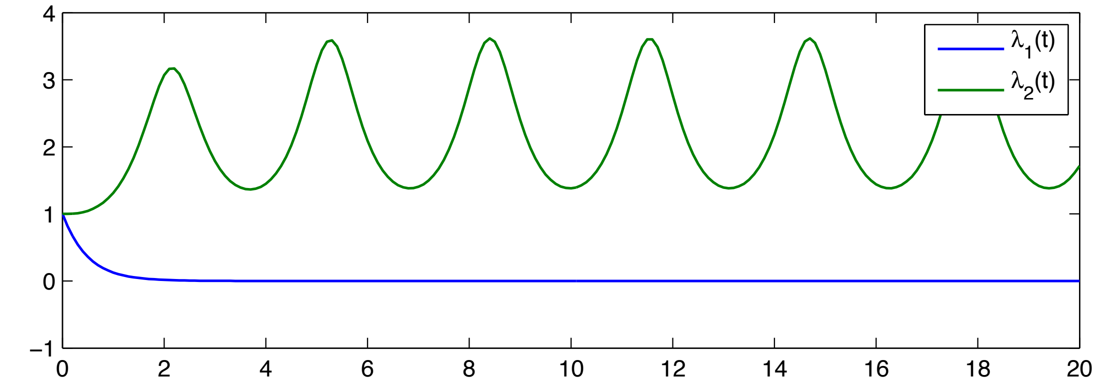


Error dynamics z(t)

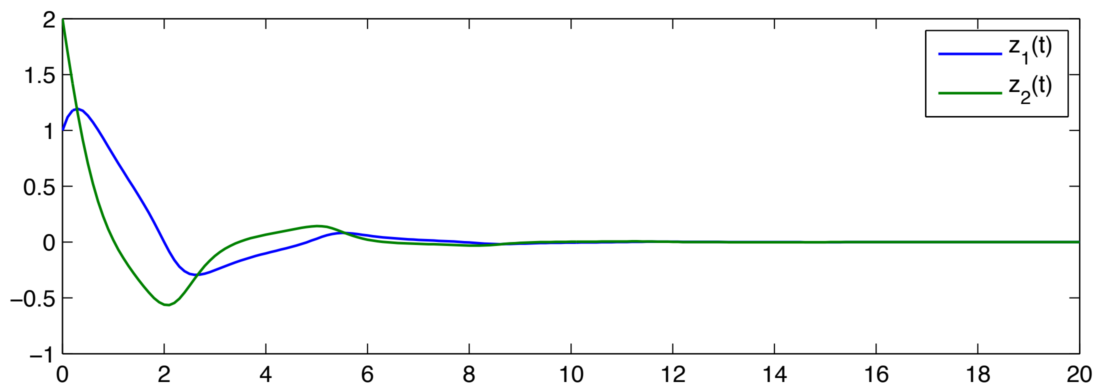
Numerical example 4

Unstable
(not anti-stable)

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix} \quad C(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad R(t) = 1.$$



Eigenvalues of P(t)



Error dynamics z(t)

Conclusion

- For output error systems, the uniform observability ensures the stability of the Kalman filter (no controllability condition).

- It is possible to design Kalman filters as if the process noise was present, but such filters are not optimal in the sense of minimum variance.