Non Lipschitz triangular canonical form for uniformly observable controlled systems

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Problem statement

We consider a dynamical system :

$$\dot{x} = f(x) + g(x)u \quad , \quad y = h(x)$$

- x : state in a compact set $\mathcal{C} \subset \mathbb{R}^n$,
- u : input in a compact set $U \subset \mathbb{R}^p$,
- y : measured output in $\mathbb{R}$.

Goal : Estimate $X(x,t)$ knowing y and u.

Definitions

Let $\mathcal{S}$ be an open set such that $\mathcal{C} \subset \mathcal{S} \subset \mathbb{R}^n$.

Definition (Uniform observability)

The system is **uniformly observable** on $\mathcal{S}$ if

for any $(x_a, x_b) \in \mathcal{S}^2$ with $x_a \neq x_b$, there is no C^1 function $u : [0, +\infty) \to U$ such that

$$h(X(x_a, t)) = h(X(x_b, t))$$

for all t such that $(X(x_a, s), X(x_b, s)) \in \mathcal{S}^2$ for all $s \leq t$.

Definition (Differential observability)

$$\mathbf{H}_m(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{m-1} h(x) \end{pmatrix}$$

$\mathbf{H}_m$ injective on $\mathcal{S}$ => **differentially observable of order** $\boldsymbol{m}$ on $\mathcal{S}$.

$\mathbf{H}_m$ injective immersion on $\mathcal{S}$ => **strongly differentially observable.**

Lipschitz triangular form

Proposition (see [1,2])

If the system is

- ***uniformly observable***
- ***strongly differentially observable** of order* $\boldsymbol{m = n}$,

$\mathbf{H}_n$ *is a the diffeomorphism and the dynamics of* $z = \mathbf{H}_n(x)$ *admit a triangular canonical form :*

$$\begin{array}{rcl} \dot{z}_1 & = & z_2 + \mathfrak{g}_1(z_1)\, u \\ & \vdots & \\ \dot{z}_i & = & z_{i+1} + \mathfrak{g}_i(z_1, \dots, z_i)\, u \\ & \vdots & \\ \dot{z}_n & = & \varphi_n(z) + \mathfrak{g}_n(z)\, u \ , \end{array} \qquad (1)$$

where the functions $\mathfrak{g}_i$ *are locally Lipschitz.*

[1] : J-P. Gauthier, G. Bornard. *Observability for any u(t) of a class of nonlinear systems*

[2] : J.-P. Gauthier, H. Hammouri, S. Othman. *A simple observer for nonlinear systems application to bioreactors*

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What if $\boldsymbol{m} > \boldsymbol{n}$?

$$\dot{\overbrace{\mathbf{H}_m(x)}} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \mathbf{H}_m(x) + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ L_f^m h(x) \end{pmatrix} + \begin{pmatrix} L_g h(x) \\ \vdots \\ L_g L_f^{i-1} h(x) \\ \vdots \\ L_g L_f^{m-1} h(x) \end{pmatrix} u$$

What if $\boldsymbol{m} > \boldsymbol{n}$?

$$\dot{\overbrace{\mathbf{H}_m(x)}} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \mathbf{H}_m(x) + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ L_f^m h(x) \end{pmatrix} + \begin{pmatrix} L_g h(x) \\ \vdots \\ L_g L_f^{i-1} h(x) \\ \vdots \\ L_g L_f^{m-1} h(x) \end{pmatrix} u$$

But do **locally Lipschitz functions** $\boldsymbol{\varphi_m}$ and $\boldsymbol{\mathfrak{g}_i}$ exist such that :

$$L_f^m h(x) = \varphi_m(h(x), \dots, L_f^{m-1} h(x))$$

$$L_g L_f^{i-1} h(x) = \mathfrak{g}_i\underbrace{(h(x), \dots, L_f^{i-1} h(x))}_{\text{triangularity}} \quad ??$$

Counter-example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^3 \\ \dot{x}_3 &= 1 + u \end{aligned} \quad , \qquad y = x_1$$

1) Uniformly observable

Counter-example

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3^3 \\ \dot{x}_3 & = & 1+u \end{array} \quad , \qquad y = x_1$$

1) Uniformly observable
2)

$$\mathbf{H}_3(x) = (x_1, x_2, x_3^3)$$

=> Differentially observable of order 3 on $\mathbb{R}^3$.

Counter-example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^3 \\ \dot{x}_3 &= 1 + u \end{aligned} \quad , \qquad y = x_1$$

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=> $\boldsymbol{\varphi_3(z_1, z_2, z_3) = 3z_3^{\frac{2}{3}}}$ **continuous but non-Lipschitz !**

Counter-example

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3^3 \\ \dot{x}_3 & = & 1+u \end{array} \quad , \qquad y = x_1$$

1) Uniformly observable
2)

$$\mathbf{H}_5(x) = (x_1, x_2, x_3^3, 3x_3^2, 6x_3)$$

=> Strongly differentially observable of order 5 on $\mathbb{R}^3$

Counter-example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^3 \quad , \qquad y = x_1 \\ \dot{x}_3 &= 1 + u \end{aligned}$$

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=> $\mathfrak{g}_3(z_1, z_2, z_3) = 3z_3^{\frac{2}{3}}$ continuous but non-Lipschitz ! !

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Continuous triangular form

Proposition

Suppose the system is ***differentially observable of order m*** *on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$.*

Then, there exists a ***continuous*** *function $\boldsymbol{\varphi_m} : \mathbb{R}^m \to \mathbb{R}$ satisfying*

$$L_f^m h(x) = \varphi_m(\boldsymbol{H}_m(x))$$

for all x in $\mathcal{C}$.

If the system is ***strongly differentially observable of order m*** *on $\mathcal{S}$, the function φ_m can be chosen* ***locally Lipschitz*** *on $\mathbb{R}^m$.*

Proposition

*Suppose the system is **uniformly observable** on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$.*

*Then, for all i, there exist **continuous** functions $\mathfrak{g}_i : \mathbb{R}^i \to \mathbb{R}$ satisfying*

$$L_g L_f^{i-1} h(x) = \mathfrak{g}_i(h(x), ..., L_f^{i-1} h(x))$$

for all x in $\mathcal{C}$.

Proposition

Suppose the system is ***uniformly observable*** *on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$.*

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Uniform observability

Proposition

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Uniform observability + differential observability

Proposition

*Suppose the system is **uniformly observable** on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$.*

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for all x in $\mathcal{C}$.

Uniform observability + differential observability

=> CONTINUOUS triangular form

Example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_3^5 x_1 \\ \dot{x}_3 &= -x_1 x_2 + u \end{aligned}$$

Example

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\dot{x}_2 &= -x_1 + x_3^5 x_1 \\
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\end{aligned}
$$

$$
\begin{aligned}
y &= x_1 \\
\dot{y} &= x_2 \\
\ddot{y} &= -x_1 + x_1 x_3^5 \\
\dddot{y} &= -x_2 + x_2 x_3^5 - 5x_3^4 x_1^2 x_2 + 5x_3^4 x_1 u
\end{aligned}
$$

=> uniformly observable on $\mathcal{S} = \left\{x \in \mathbb{R}^3 : \ x_1^2 + x_2^2 \neq 0\right\}$

Example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_3^5 x_1 \\ \dot{x}_3 &= -x_1 x_2 + u \end{aligned}$$

$$\begin{aligned} y &= x_1 \\ \dot{y} &= x_2 \\ \ddot{y} &= -x_1 + x_1 x_3^5 \\ \dddot{y} &= -x_2 + x_2 x_3^5 - 5x_3^4 x_1^2 x_2 + 5x_3^4 x_1 u \end{aligned}$$

=> uniformly observable on $\mathcal{S} = \left\{x \in \mathbb{R}^3 : \ x_1^2 + x_2^2 \neq 0\right\}$

$$\mathbf{H}_4(x) = \begin{pmatrix} x_1 \\ x_2 \\ -x_1 + x_3^5 x_1 \\ -x_2 - 5x_3^4 x_1^2 x_2 + x_3^5 x_2 \end{pmatrix}$$

=> differentially observable of order 4

=> Triangular canonical form of dimension 4

$$\begin{array}{rcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & z_3 \\ \dot{z}_3 & = & z_4 + \mathfrak{g}_3(z_1, z_2, z_3)\, u \\ \dot{z}_4 & = & \varphi_4(z) + \mathfrak{g}_4(z)\, u \\ y & = & z_1\,. \end{array} \tag{2}$$

where

- $\mathfrak{g}_3(z_1, z_2, z_3) = 5|z_3 + z_1|^{\frac{4}{5}} \lfloor z_1 \rceil^{\frac{1}{5}}$
- φ_4 and $\mathfrak{g}_4$ continuous

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Let $\mathcal{M}$ be a compact subset of $\mathbb{R}^m$ such that $\boldsymbol{H}_m(\mathcal{C}) \subset \mathcal{M}$.
Let $\mathcal{M}_i$ be the projection of $\mathcal{M}$ on $\mathbb{R}^i$.

We now look for observers for a system of the type :

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & z_2 + \Phi_1(u, z_1) \\ & \vdots & \\ \dot{z}_i & = & z_{i+1} + \Phi_i(u, z_1, \ldots, z_i) \\ & \vdots & \\ \dot{z}_m & = & \Phi_m(u, z) \end{array} \right. , \tag{3}$$

with z in $\mathcal{M}$, u in U, and Φ_i continuous on $U \times \mathcal{M}_i$.

We make the following assumption :

Assumption

For all i in $\{1, \ldots, m\}$*, there exists a compact set* $\tilde{\mathcal{M}}_i$ *strictly containing* $\mathcal{M}_i$ *such that for all* $(\mathbf{z}_{ia}, \mathbf{z}_{ib})$ *in* $\tilde{\mathcal{M}}_i^2$ *and u in U,*

$$|\Phi_i(u, \mathbf{z}_{ia}) - \Phi_i(u, \mathbf{z}_{ib})| \leq \mathfrak{a} \sum_{j=1}^{i} |z_{ja} - z_{jb}|^{\alpha_{ij}} \ . \tag{4}$$

We define :

$$\hat{\Phi}_i(u, \mathbf{z}_i) = \mathrm{sat}(\Phi_i(u, \mathbf{z}_i), \overline{\Phi}_i) \tag{5}$$

where

$$\overline{\Phi}_i = \max_{\mathbf{z}_i \in \tilde{\mathcal{M}}_i, u \in U} |\Phi_i(u, \mathbf{z}_i)|$$

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High gain observer

Consider a classical high gain observer :

$$\left\{\begin{array}{rcl} \dot{\hat{z}}_1 & = & \hat{z}_2 + \hat{\Phi}_1(u,\hat{z}_1) - L\,k_1\,(\hat{z}_1 - y) \\ \dot{\hat{z}}_2 & = & \hat{z}_3 + \hat{\Phi}_2(u,\hat{z}_1,\hat{z}_2) - L^2\,k_2\,(\hat{z}_1 - y) \\ & \vdots & \\ \dot{\hat{z}}_m & = & \hat{\Phi}_m(u,\hat{z}) - L^m\,k_m\,(\hat{z}_1 - y) \end{array}\right. \tag{6}$$

$$y = z_1$$

Proposition (Practical convergence of high gain observer)

Assume the function Φ verifies Assumption (4) with for $1 \leq j \leq i$

$$\begin{array}{c} \frac{m-i-1}{m-i} < \alpha_{ij} \leq 1 \quad \textit{for} \quad i = 1 \ldots, m-1 \ , \\ 0 \leq \alpha_{mj} \leq 1 \end{array} \tag{7}$$

Then, ***for any*** $\epsilon > \mathbf{0}$, *there exist* $\lambda > 0$, $\beta > 0$, *and* $L^* \geq 1$ *such that : for all* $L \geq L^*$, *and all* $(z, \hat{z})$ *in* $\mathcal{M} \times \mathbb{R}^m$ *such that* $Z(z,t) \in \mathcal{M}$ *for all* t *in* $\mathbb{R}_+$,

$$\left|\hat{Z}_i(\hat{z}, z, t) - Z_i(z,t)\right| \leq \max\left\{\epsilon \, , \, L^{i-1}\beta \, |\hat{z} - z| \, e^{-\lambda L t}\right\}$$

for all $t \geq 0$.

i \ j		1	2	$\dots$	$m-2$	$m-1$	m	
1		$\frac{m-2}{m-1}$						
2		$\frac{m-3}{m-2}$	$\frac{m-3}{m-2}$					
$\vdots$	$\alpha_{ij} >$	$\vdots$	$\vdots$	$\ddots$				
$m-2$		$\frac{1}{2}$	$\frac{1}{2}$	$\dots$	$\frac{1}{2}$			
$m-1$		0	0	$\dots$	$\dots$	0		$\leftarrow$ "Hölder"
m	$\alpha_{mj} \geq$	0	0	$\dots$	$\dots$	$\dots$	0	$\leftarrow$ "bounded"

Table 1 : Hölder restrictions on Φ for arbitrarily small errors with a high gain observer.

Example

$$\begin{array}{rcl} \dot z_1 &=& z_2 \\ \dot z_2 &=& z_3 \\ \dot z_3 &=& z_4 + \Phi_3(u, z_1, z_2, z_3) \\ \dot z_4 &=& \Phi_4(u, z) \\ y &=& z_1 \,. \end{array} \quad , \quad \Phi_3(u, z_1, z_2, z_3) = 5u|z_3+z_1|^{\frac{4}{5}} \lfloor z_1 \rceil^{\frac{1}{5}} \,.$$

Φ_4 is continuous => bounded on $U \times \mathcal{M}_4$

There exist c_1 and c_3 such that on $U \times \mathcal{M}_3$

$$|\Phi_3(u, \hat z_1, \hat z_2, \hat z_3) - \Phi_3(u, z_1, z_2, z_3)| \leq c_1 |\hat z_1 - z_1|^{\frac{1}{5}} + c_3 |\hat z_3 - z_3|^{\frac{4}{5}}$$

=> $\alpha_{31} = \frac{1}{5} > 0$, $\alpha_{32} = \frac{4}{5} > 0$

=> practical convergence of high gain observer

L	$e_{z,1}$	$e_{z,2}$	$e_{z,3}$	$e_{z,4}$	Peaking
2	0.15	4	60	200	300
5	$6\,.\,10^{-4}$	0.04	1.5	30	4000
8	$5\,.\,10^{-5}$	$4\,.\,10^{-3}$	0.25	7	$1.5\,.\,10^{4}$
10	$8\,.\,10^{-6}$	$1\,.\,10^{-3}$	0.1	4	$3.5\,.\,10^{4}$
15	$1.5\,.\,10^{-6}$	$3\,.\,10^{-4}$	0.03	2	$1.2\,.\,10^{5}$

Table: Decrease of the final error in the z-coordinates $(e_{z,i} = \hat{z}_i - z_i)$ with the gain L, with a high gain observer.

=> BUT, the larger L, the larger the peaking and the noise amplification...

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Homogeneous observer

Consider the homogeneous observer [1 2] :

$$\begin{cases} \dot{\hat{z}}_1 & = & \hat{z}_2 + \hat{\Phi}_1(u, \hat{z}_1) - L\, k_1 \, \lfloor \hat{z}_1 - y \rceil^{\frac{r_2}{r_1}} \\ \dot{\hat{z}}_2 & = & \hat{z}_3 + \hat{\Phi}_2(u, \hat{z}_1, \hat{z}_2) - L^2\, k_2 \, \lfloor \hat{z}_1 - y \rceil^{\frac{r_3}{r_1}} \\ & \vdots & \\ \dot{\hat{z}}_m & = & \hat{\Phi}_m(u, \hat{z}) - L^m\, k_m \, \lfloor \hat{z}_1 - y \rceil^{\frac{r_{m+1}}{r_1}} \end{cases} \tag{8}$$

where $\lfloor a \rceil^p = \text{sign}(a)|a|^p$,

$$r_i = 1 - d_0(m - i)$$

with d_0 in $[-1, 0]$.

1. V. Andrieu, L. Praly, A. Astolfi. *Homogeneous approximation, recursive observer design, and output feedback.*
2. C. Qian. *A homogeneous domination approach for global output feedback stabilization of a class of nonlinear systems*

Particular case [3] : $d_0 = -1$

$$\left\{\begin{array}{rcl} \dot{\hat{z}}_1 & = & \hat{z}_2 + \hat{\Phi}_1(u,\hat{z}_1) - L\,k_1\,\lceil \hat{z}_1 - y \rfloor^{\frac{m-1}{m}} \\ \dot{\hat{z}}_2 & = & \hat{z}_3 + \hat{\Phi}_2(u,\hat{z}_1,\hat{z}_2) - L^2\,k_2\,\lceil \hat{z}_1 - y \rfloor^{\frac{m-2}{m}} \\ & \vdots & \\ \dot{\hat{z}}_{m-1} & = & \hat{z}_m + \hat{\Phi}_{m-1}(u,\hat{z}_1,\ldots,\hat{z}_{m-1}) - L^{m-1}\,k_{m-1}\,\lceil \hat{z}_1 - y \rfloor^{\frac{1}{m}} \\ \dot{\hat{z}}_m & \in & \hat{\Phi}_m(u,\hat{z}) - L^m\,k_m\,\mathrm{S}(\hat{z}_1 - y) \end{array}\right. \tag{9}$$

with

$$\mathrm{S}(a) = \begin{cases} \{1\} & \text{if } a > 0\,, \\ [-1,1] & \text{if } a = 0\,, \\ \{-1\} & \text{if } a < 0\,. \end{cases}$$

3. A. Levant. *Higher-order sliding modes and arbitrary-order exact robust differentiation*

Proposition (Finite time convergence of homogeneous observer)

Assume there exist d_0 in $[-1,0)$ such that Φ verifies Assumption (4) with with

$$\alpha_{ij} \geq \frac{1-d_0(m-i-1)}{1-d_0(m-j)} \ .$$

Then, there exist $(k_1,\dots,k_m)$ and $L^* \geq 1$ such that :
for all $L \geq L^*$, all $(z,\hat{z})$ in $\mathcal{M}\times\mathbb{R}^m$ such that $Z(z,t)\in\mathcal{M}$ for all t in $\mathbb{R}_+$, there exists T, depending on $(\hat{z},z)$, such that :

$$\hat{Z}(\hat{z},z,t) \ = \ Z(z,t) \ , \ \forall t \geq T \ .$$

i \ j		1	2	$\dots$	$m-2$	$m-1$	m
1		$\frac{m-1}{m}$					
2		$\frac{m-2}{m}$	$\frac{m-2}{m-1}$				
$\vdots$	$\alpha_{ij} \geq$	$\vdots$	$\vdots$	$\ddots$			
$m-2$		$\frac{2}{m}$	$\frac{2}{m-1}$	$\dots$	$\frac{2}{3}$		
$m-1$		$\frac{1}{m}$	$\frac{1}{m-1}$	$\dots$	$\dots$	$\frac{1}{2}$	
m		0	0	$\dots$	$\dots$	$\dots$	0

Table 3 : Hölder restrictions on Φ for a homogeneous observer with $d_0 = -1$

=> What if those restrictions are not satisfied ?

Example

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 + \Phi_3(u, z_1, z_2, z_3) \quad , \quad \Phi_3(u, z_1, z_2, z_3) = 5u|z_3+z_1|^{\frac{4}{5}} \lfloor z_1 \rceil^{\frac{1}{5}} . \\ \dot{z}_4 &= \Phi_4(u, z) \\ y &= z_1 . \end{aligned}$$

$$|\Phi_3(u, \hat{z}_1, \hat{z}_2, \hat{z}_3) - \Phi_3(u, z_1, z_2, z_3)| \leq c_1 |\hat{z}_1 - z_1|^{\frac{1}{5}} + c_3 |\hat{z}_3 - z_3|^{\frac{4}{5}}$$

=> $\alpha_{31} = \frac{1}{5} < \frac{1}{4}$, $\alpha_{32} = \frac{4}{5} > \frac{1}{2}$

=> no theoretical guarantee of convergence of homogeneous observer

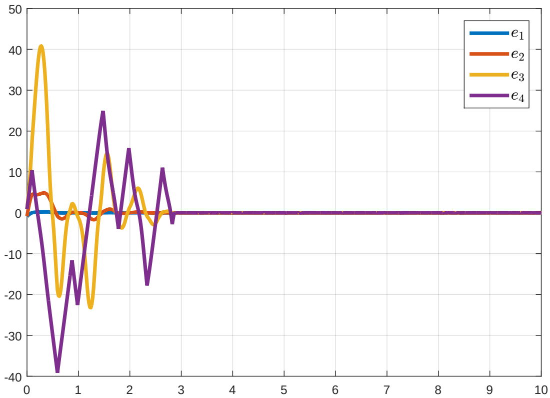

Figure: Error in the z-coordinates with a homogeneous observer

Cascaded homogeneous observer

$$
\begin{aligned}
\dot{\hat{z}}_{11} &\in -L_1\, k_{11}\, \mathrm{S}(\hat{z}_{11} - y) \\
&\cdots\cdots\cdots \\
\dot{\hat{z}}_{21} &= \hat{z}_{22} + \hat{\Phi}_1(u, \hat{z}_{11}) - L_2\, k_{21}\, \lfloor \hat{z}_{21} - y \rceil^{\frac{1}{2}} \\
\dot{\hat{z}}_{22} &\in -L_2^2\, k_{22}\, \mathrm{S}(\hat{z}_{21} - y) \\
&\cdots\cdots\cdots \\
&\vdots \\
&\cdots\cdots\cdots \\
\dot{\hat{z}}_{m1} &= \hat{z}_{m2} + \hat{\Phi}_1(u, \hat{z}_{(m-1)1}) - L_m\, k_{m1}\, \lfloor \hat{z}_{m1} - y \rceil^{\frac{m-1}{m}} \\
\dot{\hat{z}}_{m2} &= \hat{z}_{m3} + \hat{\Phi}_2(u, \hat{z}_{(m-1)1}, \hat{z}_{(m-1)2}) - L_m^2\, k_{m2}\, \lfloor \hat{z}_{m1} - y \rceil^{\frac{m-2}{m}} \\
&\vdots \\
\dot{\hat{z}}_{m(m-1)} &= \hat{z}_{mm} + \hat{\Phi}_{m-1}(u, \hat{z}_{(m-1)1}, \ldots, \hat{z}_{(m-1)(m-1)}) \\
&\qquad - L_m^{m-1}\, k_{m(m-1)}\, \lfloor \hat{z}_{m1} - y \rceil^{\frac{1}{m}} \\
\dot{\hat{z}}_{mm} &\in -L_m^m\, k_{mm}\, \mathrm{S}(\hat{z}_{m1} - z_1)
\end{aligned}
\qquad (10)
$$

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Proposition (Finite-time convergence of cascaded homogeneous observer)

*Assume Φ is **locally bounded**.*

Then, there exist $k_{ij} > 0$ and $L^* > 0$ such that :
for all $L \geq L^*$ and all $(z, \hat{z})$ in $\mathcal{M} \times \mathbb{R}^{\frac{m(m+1)}{2}}$ such that $Z(z,t) \in \mathcal{M}$ for all t in $\mathbb{R}_+$, there exists T such that

$$\hat{\mathbf{Z}}_m(\hat{z}, z, t) \;=\; Z(z,t) \qquad \forall t \geq T \ .$$

where $\hat{\mathbf{Z}}_m$ denotes the last block of m components of $\hat{Z}$.

$+$: no constraint on Φ

$-$: high dimension (but not necessarily $\frac{m(m+1)}{2}$)

Example

$$
\begin{aligned}
\dot{\hat{z}}_{11} &= \hat{z}_{12} - L_1\, k_{11}\, \lfloor \hat{z}_{11} - y \rceil^{\frac{2}{3}} \\
\dot{\hat{z}}_{12} &= \hat{z}_{13} - L_1^2\, k_{12}\, \lfloor \hat{z}_{11} - y \rceil^{\frac{1}{3}} \\
\dot{\hat{z}}_{13} &\in -L_1^3\, k_{13}\, \mathrm{S}(\hat{z}_{11} - y)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\hat{z}}_{21} &= \hat{z}_{22} - L_2\, k_{21}\, \lfloor \hat{z}_{21} - y \rceil^{\frac{3}{4}} \\
\dot{\hat{z}}_{22} &= \hat{z}_{23} - L_2^2\, k_{22}\, \lfloor \hat{z}_{21} - y \rceil^{\frac{1}{2}} \\
\dot{\hat{z}}_{23} &= \hat{z}_{24} + \mathrm{sat}_3(\mathfrak{g}_3(\hat{z}_{11}, \hat{z}_{12}, \hat{z}_{13}))u - L_2^3\, k_{23}\, \lfloor \hat{z}_{21} - y \rceil^{\frac{1}{4}} \\
\dot{\hat{z}}_{24} &\in -L_2^4\, k_{24}\, \mathrm{S}(\hat{z}_{21} - y)
\end{aligned}
$$

Contents

Conclusion

- uniform observability + differential observability
  = continuous triangular form
- high gain observer + conditions in Table 1 -> practical convergence
  -> problem of peaking and noise amplification
- homogeneous observer + conditions in Table 2 -> finite time convergence
- cascade of homogeneous observers -> finite time convergence
  -> high dimension but better tolerance to noise ?
- for no restriction on compact sets, homogeneity in the bi-limit [4]

4. V. Andrieu, L. Praly, A. Astolfi. *Homogeneous approximation, recursive observer design, and output feedback.*