Parameter and state estimation of nonlinear systems using a supervisory observer

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joint work with Michelle Chong[2], Dragan Nešić[3] and Levin Kuhlmann[3]

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Motivation

System

$$\begin{cases} \dot{x} &= f(x, p^\star, u) \\ y &= h(x, p^\star) \end{cases}$$

- $x \in \mathbb{R}^{n_x}$ state
- $p^\star \in \mathbb{R}^{n_p}$ unknown parameters
- $u \in \mathbb{R}^{n_u}$ input
- $y \in \mathbb{R}^{n_y}$ measured output

Objective

To estimate on-line x and $p^\star$

Application: neurosciences (neural mass models)

Existing approaches

- Parameters as states
  - ✓ easy to do
  - ✗ may destroy interesting properties of f w.r.t. x
- Adaptive observers[1]
  - ✓ stable
  - ✗ specific classes of systems
  - ✗ may not be robust to large uncertainties

[1] [Kresselmeier, IEEE TAC 1977; Marino and Tomei, IEEE TAC 1992; Zhang IEEE TAC 2002; Farza et al. Automatica 2009; etc.]

A detour to control

- Adaptive control, i.e. $u = k(x, \hat{p})$ and $\dot{\hat{p}} = g(x, \hat{p})$
  - ✓ stable
  - ✗ may not be robust to large uncertainties
- Multiple model [Narendra and Han, Annual Reviews in Control 2011], i.e. $u = k(x, p_i)$ and $p_i \in \Theta_i$ with Θ_i discrete finite set
  - ✓ robustness
  - ✗ may be computationally demanding

Idea

To adapt the multiple-model approach to nonlinear estimation
↪ inspired by "supervisory control" works in [Morse IEEE TAC 1997; Hespanha et al, Automatica 2003; Vu and Liberzon IEEE TAC 2011; etc.]

Existing works: linear systems and different approaches [Aguiar et al., ECC 2007, IFAC WC 2007; Li and Bar-Shalom IEEE TAC 1996; etc.]

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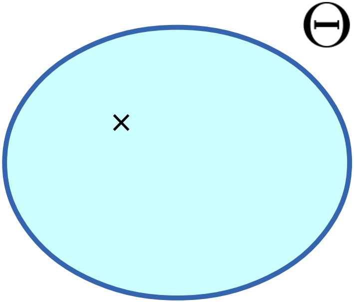
Main idea

Assumption: $p^\star \in \Theta$ with Θ a **known** compact set

1. State observer for any **known** $p^\star \in \Theta$
2. Discretization of Θ with N points
   $\rightarrow$ N associated state observers
3. Selection criterion
4. Convergence analysis for large N

\+ Dynamic discretization of Θ



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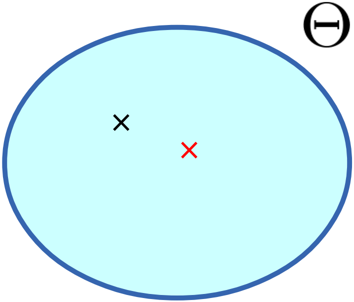

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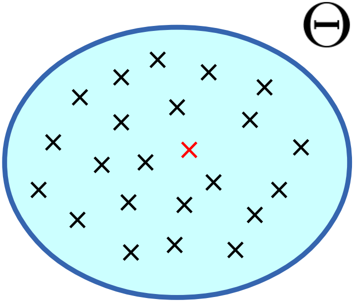

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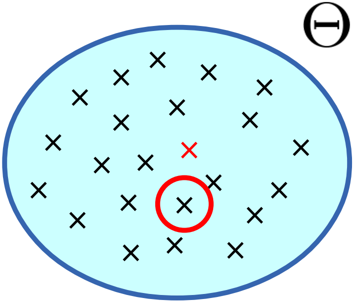

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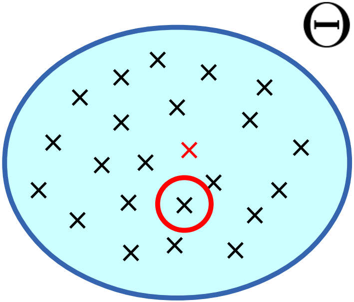

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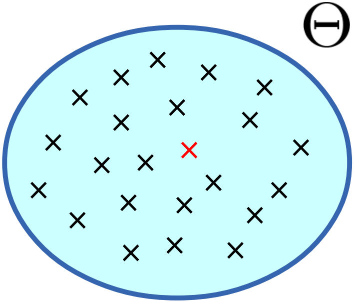

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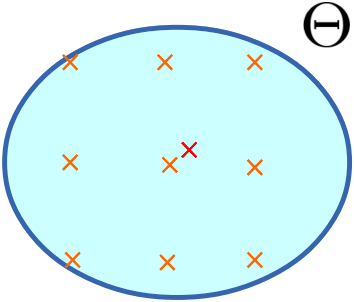

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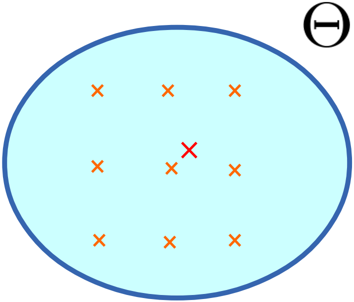

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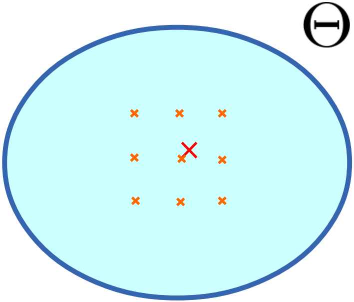

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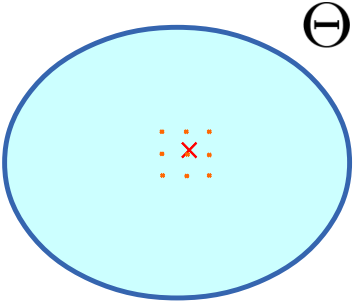
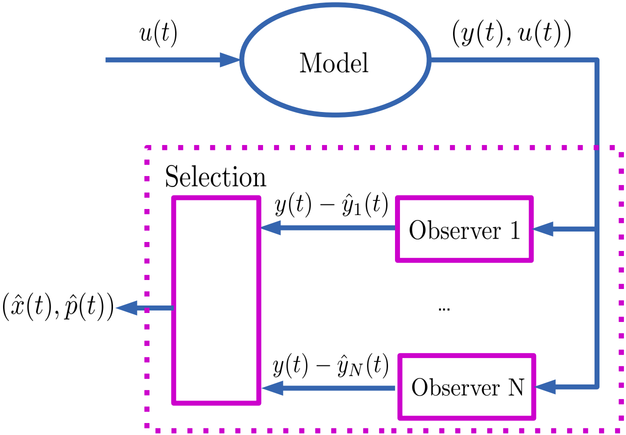


Scheme: supervisory observer



Overview

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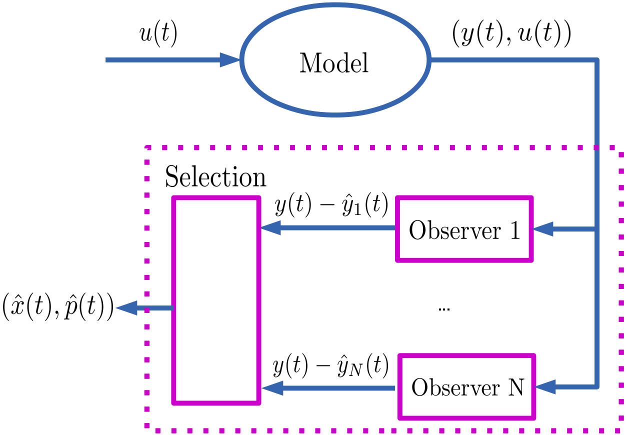
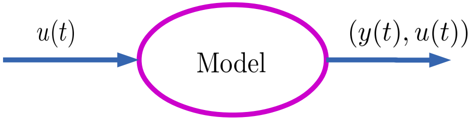
Scheme



$u(t)$
Model
$(y(t), u(t))$
Selection
$y(t) - \hat{y}_1(t)$
Observer 1
...
$(\hat{x}(t), \hat{p}(t))$
$y(t) - \hat{y}_N(t)$
Observer N

Preliminary assumption on the model



Preliminary assumption on the model (c'd)

Assumption: bounded system

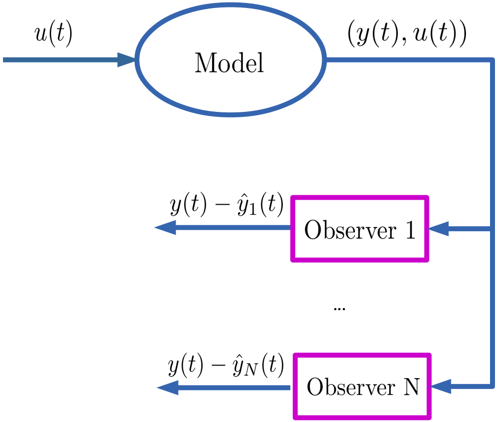
The solutions to $\dot{x} = f(x, p^\star, u)$ are uniformly bounded, i.e.

- for any radius $\Delta_x > 0$ of the hypercube of initial conditions
- for any $\Delta_u > 0$ bound on the input u

there exists $K \geq 0$ such that $\forall x(0)$ with $|x(0)|_\infty < \Delta_x$, and piecewise continuous u with $|u(t)|_\infty \leq \Delta_u$ for any $t \geq 0$,

$$|x(t)|_\infty \leq K \quad \forall t \geq 0$$

State-observer



State-observer (c'd)

System

$$\begin{cases} \dot{x} = f(x, p^\star, u) \\ y = h(x, p^\star) \end{cases}$$

State observer, for any $p \in \Theta$,

$$\begin{cases} \dot{\hat{x}} = \hat{f}(\hat{x}, p, u, y) \\ \hat{y} = h(\hat{x}, p) \end{cases}$$

Let $\tilde{x} := x - \hat{x}$ and $\tilde{p} := p^\star - p$

Assumption: robust state-observer

There exist

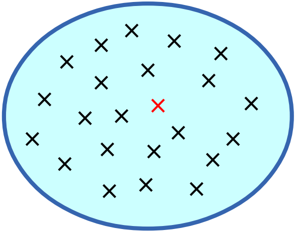
- $V_p \in C^1(\mathbb{R}^{n_x}, \mathbb{R}_{\geq 0})$, for any $p \in \Theta$
- $a_1, a_2, \lambda > 0$
- $\gamma : \mathbb{R}^{n_p+n_x+n_u} \to \mathbb{R}_{\geq 0}$ continuous such that $\gamma(0, x, u) = 0$

such that for all $x, \hat{x}, p, u$

$$\begin{aligned} a_1|\tilde{x}|_\infty^2 &\leq V_p(\tilde{x}) \leq a_2|\tilde{x}|_\infty^2 \\ \left\langle \nabla V_p(\tilde{x}), f(x, p^\star, u) - \hat{f}(\hat{x}, p, u, y) \right\rangle &\leq -\lambda V_p(\tilde{x}) + \gamma(\tilde{p}, x, u) \end{aligned}$$

Discretization of Θ

Let $N \in \mathbb{Z}_{>0}$ and $p_i \in \Theta$, $i \in \{1, \ldots, N\} \longrightarrow \widehat{\Theta}_N = \{p_1, \ldots, p_N\}$



Property of the discretization:

$$\min_{p \in \widehat{\Theta}_N} |p^\star - p|_\infty \to 0 \text{ as } N \to \infty$$

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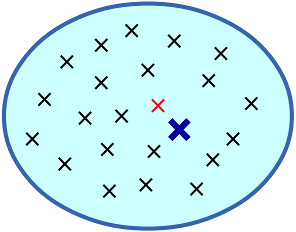

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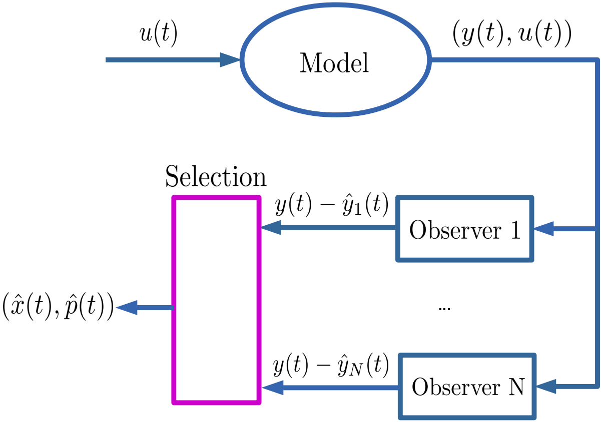
Selection criterion



Selection criterion (c'd)

Monitoring signals, for $i \in \{1, \dots, N\}$,

$$\mu_i(t) = \int_0^t \exp(-\lambda(t-s))|y(s) - \hat{y}_i(s)|_\infty^2 \, \mathrm{d}s \quad \forall t \geq 0$$

where $\lambda > 0$ is a design parameter

Implementation

$$\begin{aligned} \dot{\mu}_i(t) &= -\lambda \mu_i(t) + |\tilde{y}_i(t)|_\infty^2 \qquad \forall t \geq 0 \\ \mu_i(0) &= 0 \end{aligned}$$

Selection criterion (c'd)

Recall, for $i \in \{1, \ldots, N\}$,

$$\mu_i(t) = \int_0^t \exp(-\lambda(t-s))|y(s) - \hat{y}_i(s)|_\infty^2 \, \mathrm{d}s \quad \forall t \geq 0$$

Assumption: persistency of excitation

- for any radius $\Delta_x, \Delta_{\tilde{x}} > 0$ of the hypercubes of initial conditions
- for any $\Delta_u > 0$ bound on the input u

there exists $\alpha \in \mathcal{K}_\infty$ such that $\forall p_i \in \Theta$, $x(0), \tilde{x}_i(0)$ with $|\tilde{x}_i(0)|_\infty < \Delta_{\tilde{x}}$, $|x(0)|_\infty < \Delta_x$ and some piecewise continuous u with $|u(t)|_\infty < \Delta_u$ for all $t \geq 0$,

$$\int_{t-T_f}^t |y(s) - \hat{y}_i(s)|_\infty^2 \, \mathrm{d}s \geq \alpha(|p^\star - p_i|_\infty) \qquad \forall t \geq T_f.$$

Assumption: persistency of excitation (c'd)

If

- the boundedness assumption on the plant holds
- f and h are C^1
- classical PE holds

then the previous persistency of excitation condition is satisfied

Selection criterion (c'd)

Selection law

$$\sigma(t) \in \underset{i \in \{1,\ldots,N\}}{\arg\min} \mu_i(t) \qquad \forall t \geq 0$$

Parameter and state estimates

$$\left\{ \begin{array}{rcl} \hat{p}(t) & = & p_{\sigma(t)}(t) \\ \hat{x}(t) & = & x_{\sigma(t)}(t) \end{array} \right. \qquad \forall t \geq 0$$

Convergence guarantees

Theorem

Under the previous assumptions

- for any desired margin of errors $\nu_{\tilde{x}}, \nu_{\tilde{p}} > 0$
- for any radius $\Delta_x, \Delta_{\tilde{x}} > 0$ of the hypercubes of initial conditions
- for any $\Delta_u > 0$ bound on the input u

there exist

- a sufficiently long time $T > 0$
- and a sufficiently big number of observers $N^\star$

such that $\forall N \geq N^\star$, $\forall x(0), \tilde{x}_i(0)$ with $|x(0)|_\infty \leq \Delta_x$ and $|\tilde{x}_i(0)|_\infty \leq \Delta_{\tilde{x}}$ and any piecewise continuous u with $|u(t)|_\infty \leq \Delta_u$

$$|\tilde{p}_{\sigma(t)}(t)|_\infty \leq \nu_{\tilde{p}} \qquad \forall t \geq T$$

$$\limsup_{t\to\infty} |\tilde{x}_{\sigma(t)}(t)|_\infty \leq \nu_{\tilde{x}}$$

Convergence guarantees: main elements of the proof

- ISS property of each estimation error system, $i \in \{1,\dots,N\}$,

$$|\tilde{x}_i(t)|_\infty \le \bar{k}\exp(-\bar{\lambda}t)|\tilde{x}_i(0)|_\infty + \bar{\gamma}_{\tilde{x}}(|\tilde{p}_i|_\infty), \qquad \forall t \ge 0$$

- For any $\epsilon > 0$, there exist $\underline{\chi}, \overline{\chi} \in \mathcal{K}_\infty$ such that

$$\underline{\chi}(|\tilde{p}_i|_\infty) \le \mu_i(t) \le \bar{\chi}(|\tilde{p}_i|_\infty) + \epsilon \qquad t \ge T$$

(recall that $\mu_i(t) = \int_0^t \exp(-\lambda(t-s))|y(s) - \hat{y}_i(s)|_\infty^2 \, ds$)

- Since $\sigma(t) \in \arg\min_{i\in\{1,\dots,N\}} \mu_i(t)$, we have $\mu_{\sigma(t)} \le \mu_i(t)$ for any $i \in \{1,\dots,N\}$ and $t \ge 0$
  $\hookrightarrow \mu_{\sigma(t)} \le \mu_{i^\star}(t) \le \overline{\chi}(|\tilde{p}_{i^\star}|_\infty) + \epsilon$
  $\hookrightarrow \mu_{\sigma(t)}$ as small as desired

- This implies $\underline{\chi}(|\tilde{p}_{\sigma(t)}|_\infty)$ as small as desired $\rightarrow$ parameter estimation error property ✓

- Hence

$$|\tilde{x}_{\sigma(t)}(t)|_\infty \le \bar{k}\exp(-\bar{\lambda}t)|\tilde{x}_{\sigma(t)}(0)|_\infty + \bar{\gamma}_{\tilde{x}}(|\tilde{p}_{\sigma(t)}|_\infty), \qquad \forall t \ge 0$$

$\hookrightarrow$ properties on the state estimation error follows ✓

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Dynamic discretization of Θ: the idea

For any $\nu_{\tilde{p}}$, for N large

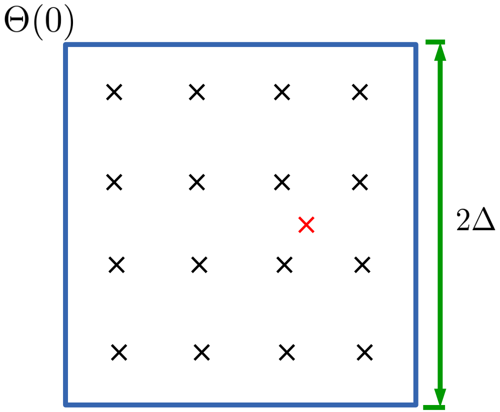
$$|\tilde{p}_{\sigma(t)}(t)|_\infty = |p^\star - p_{\sigma(t)}(t)|_\infty \leq \nu_{\tilde{p}} \quad \forall t \geq T$$

Idea

Redistribute the p_i's within hypercube centered at $p_{\sigma(t)}$ of radius $\nu_{\tilde{p}}$ when $t \geq T$

Dynamic discretization of Θ: the idea (c'd)

$\Theta = \Theta(0) = \mathcal{H}(0, \Delta)$,



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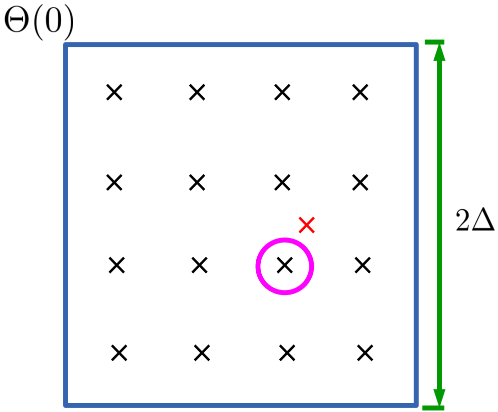

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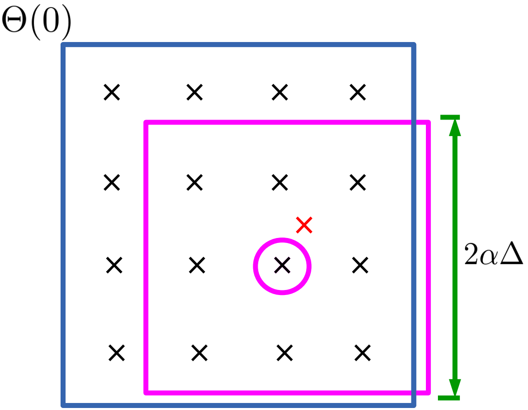

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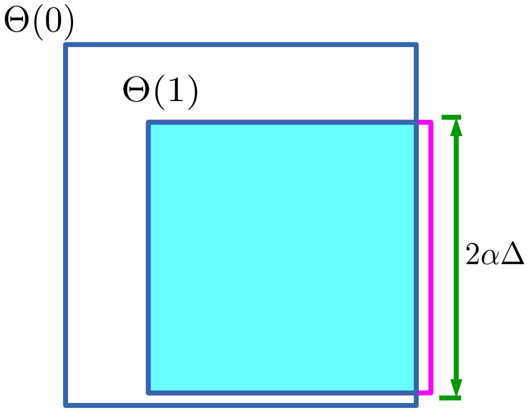

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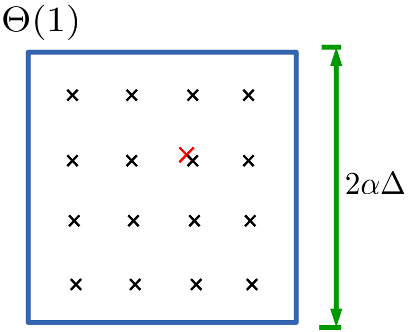

Dynamic discretization of Θ: the update rule

Assumption

Θ is an hypercube $\mathcal{H}(p_c, \Delta)$, $\Delta > 0$

Let t_k, $k \in \mathbb{Z}_{>0}$ be such that, for any $k \in \mathbb{Z}_{>0}$

$$t_{k+1} - t_k \geq T$$

where $T > 0$ design parameter

- $t_0 = 0$, $\Delta(0) = \Delta$ and $\Theta(0) = \Theta$
- $t = t_k$, $k \in \mathbb{Z}_{>0}$,

$$\Delta(k) = \alpha\Delta(k-1)$$

where $\alpha \in (0,1)$ zooming factor

$$\Theta(k) = \mathcal{H}(\hat{p}(t_k^-), \Delta(k)) \cap \Theta(k-1) \cap \ldots \cap \Theta(0)$$

Dynamic discretization of Θ: convergence guarantees

Theorem

Under the previous assumptions

- for any desired margin of errors $\nu_{\tilde{x}}, \nu_{\tilde{p}} > 0$
- for any radius $\Delta_x, \Delta_{\tilde{x}} > 0$ of the hypercubes of initial conditions
- for any $\Delta_u > 0$ bound on the input u
- for any zooming factor $\alpha \in (0, 1)$

there exist

- a sufficiently zoom-in period $T^\star > 0$
- a sufficiently long time $T > 0$
- a sufficiently big number of observers $N^\star$

such that $\forall N \geq N^\star$, $\forall T > T^\star$, $\forall x(0), \tilde{x}_i(0)$ with $|x(0)|_\infty \leq \Delta_x$ and $|\tilde{x}_i(0)|_\infty \leq \Delta_{\tilde{x}}$ and any piecewise continuous u with $|u(t)|_\infty \leq \Delta_u$

$$|\tilde{p}_{\sigma(t)}(t)|_\infty \leq \nu_{\tilde{p}} \qquad \forall t \geq T$$

$$\limsup_{t \to \infty} |\tilde{x}_{\sigma(t)}(t)|_\infty \leq \nu_{\tilde{x}}$$

Overview

Applications: case studies

- Linear systems

$$\begin{array}{rcl} \dot{x} & = & A(p^\star)x + B(p^\star)u \\ y & = & C(p^\star)x \end{array}$$

- A class of nonlinear systems

$$\begin{array}{rcl} \dot{x} & = & A(p^\star)x + G(p^\star)\widetilde{\gamma}(Hx) + B(p^\star)\phi(u,y) \\ y & = & C(p^\star)x \end{array}$$

where $\widetilde{\gamma}$ is slope-restricted

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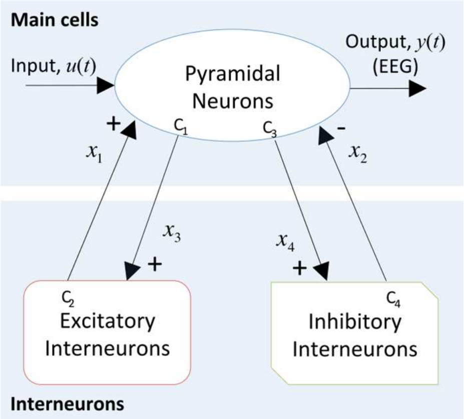
Applications: Jansen & Rit's model



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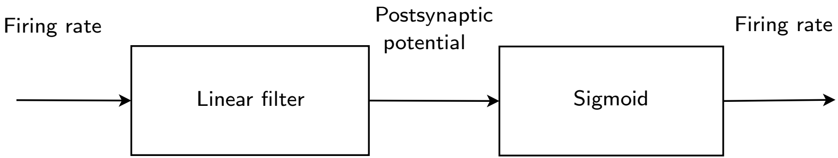

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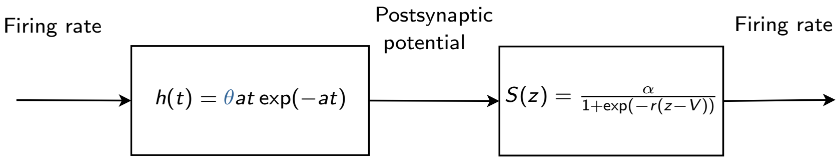

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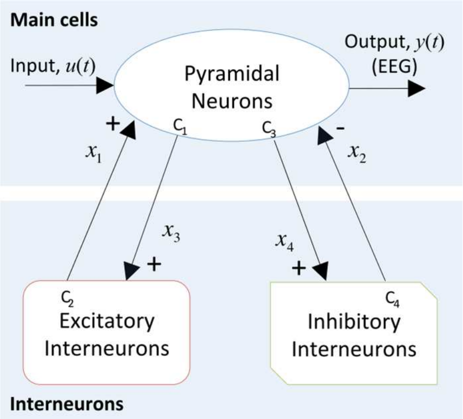

Applications: Jansen & Rit's model (c'd)

$$
\begin{aligned}
\dot{z}_0 &= z_3\\
\dot{z}_3 &= -2az_3 - a^2 z_0 + p_1 a S(z_1 - z_2)\\
\dot{z}_1 &= z_4\\
\dot{z}_4 &= -2az_4 - a^2 z_1 + p_1 a\,(u_i + c_2 S(c_1 z_0))\\
\dot{z}_2 &= z_5\\
\dot{z}_5 &= -2bz_5 - b^2 z_2 + p_2 b c_4 S(c_3 z_0)\\
y &= z_1 - z_2
\end{aligned}
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1 output / 6 state variables + 2 parameters

$p_1 \in [4, 8]$, $p_2 \in [22, 28]$

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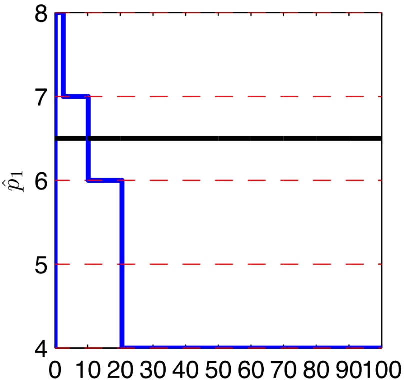
Applications: Jansen & Rit's model (c'd)

Static sampling with $N = 5 \times 5$



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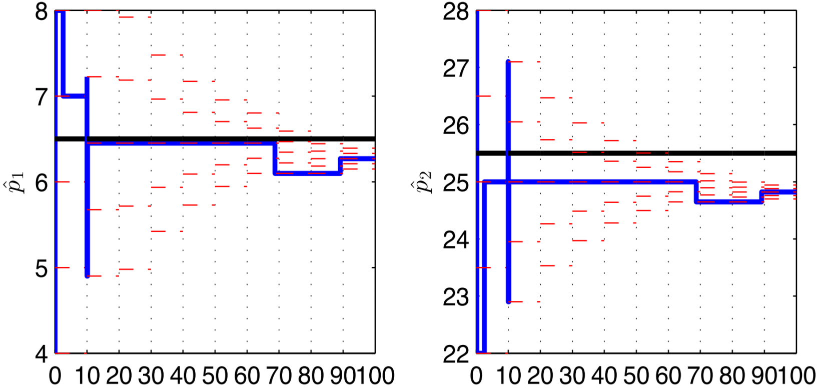

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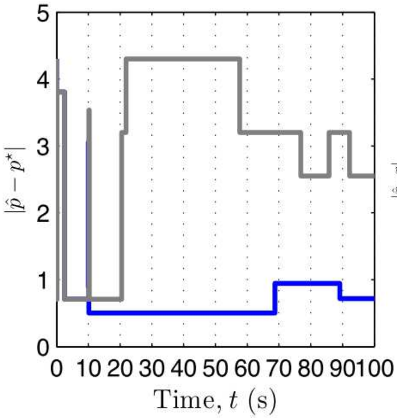

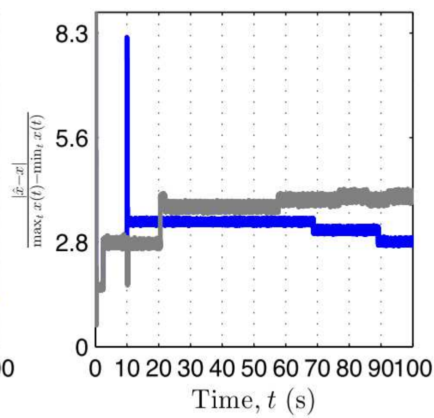

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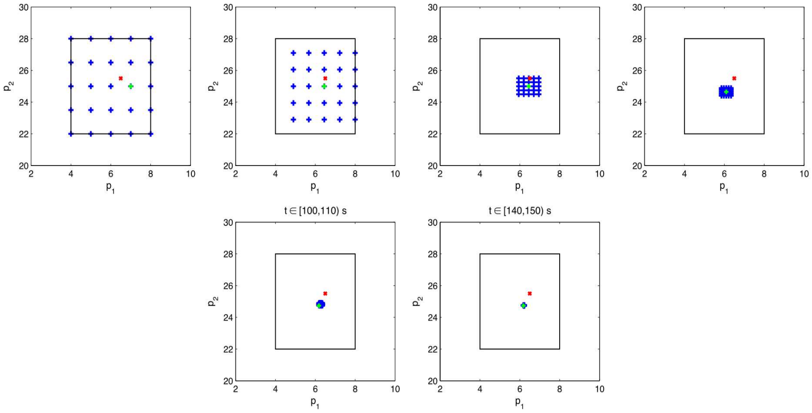

Applications: Jansen & Rit's model (c'd)

$N = N_A \times N_B$	2×2	4×4	5×5
Static policy: $\left\|\tilde{p}_{\sigma(t_f)}(t_f)\right\|$	4.30	3.80	2.55
Dynamic policy: $\left\|\tilde{p}_{\sigma(t_f)}(t_f)\right\|$	2.66	1.04	0.72
Static policy: $\frac{\left\|\tilde{x}_{\sigma(t_f)}(t_f)\right\|}{\max_t \|x(t)\| - \min_t \|x(t)\|}$	4.01	1.73	4.64
Dynamic policy: $\frac{\left\|\tilde{x}_{\sigma(t_f)}(t_f)\right\|}{\max_{t\in[0,t_f]} \|x(t)\| - \min_{t\in[0,t_f]} \|x(t)\|}$	4.26	3.47	3.22

The final time of simulation t_f is 100s.

Table: Numerical results for increasing number of observers N such that $d(p^\star, \widehat{\Theta})$ decreases in the static case.

Overview

1. Introduction
2. Supervisory observer
3. Applications
4. Conclusions

Conclusions: summary

Summary

- Deterministic estimation algorithm for nonlinear systems
- Dynamic discretization policy to reduce computation complexity
- Application to a neural mass model

Reference: M. Chong et al., Parameter and state estimation of nonlinear systems using a multi-observer under the supervisory framework, *IEEE Transactions on Automatic Control*, 60(9), 2336-2349, 2015.